Stability of Networked Control Systems with Time-Varying Transmission Period

Wei Zhang and Michael S. Branicky*
Electrical Engineering and Computer Science
Case Western Reserve University

Abstract

Feedback control systems wherein the control loops are closed through a real-time network are called networked control systems (NCSs). Constant transmission period cannot be guaranteed in an NCS because transmissions have to be scheduled among other NCS plants on the network. We study the stability of NCSs with time-varying transmission period. We derive sufficient conditions on the transmission period that guarantee the NCS will be stable. We also discuss methods to numerically search for such bounds on the transmission period.

1 Introduction

This paper studies the stability of NCSs with time-varying transmission period. The NCS *transmission period*, denoted by $h(t)$, is defined as the time interval between two consecutive transmissions, measured between the instants when the transmissions arrive. Ignoring the data processing overhead including building and queueing the data packet, the terms *transmission period* and *sampling period* of the sensor(s) can be used as synonyms. Data transmissions on NCSs are effected by many nondeterministic factors, such as jitters, delays, and transient errors [6]. Equal-distance sampling cannot be guaranteed on NCSs, hence it is important to analyze the stability of NCSs when the sampling period (or transmission period) is varying.

Figure 1 illustrates the system under study in this paper. Sampling of the plant state is taken at $t_k, k = 0, 1, \ldots$, however the time intervals between the sampling instants are not constant. Furthermore, we only consider the case where the sensor-to-controller path is networked, as shown in Figure 1.

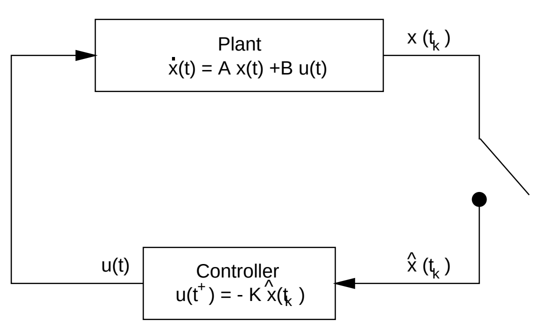

Figure 1: System model under consideration

*This work partially supported by NSF grant DMI97-20309.

For this paper, we assume the transmission process is ideal. In other words, we assume that there is no data processing time, scheduling time, or transmission time. Hence, the sensors' data is delivered instantaneously without any delay. We further assume that there is no data loss (packet loss) during the transmission. We have relaxed these assumptions on the transmission process in other work. Specifically, network scheduling, network-induced delay, and packet dropout were studied in [7].

Throughout this paper, we assume that the control system can be designed without having the network in mind. That is, the original continuous plant plus the continuous state feedback controller without the network connection is stable or satisfies certain control specifications. In our setup, we consider (a) clock-driven sensors that sample the plant outputs periodically or aperiodically at sampling instants; (b) an event-driven controller, which can be implemented by an external event interrupt mechanism and which calculates the control signal as soon as the sensor data arrives; and (c) event-driven actuators, which means the plant inputs are changed as soon as the data become available.

Our goal in this paper is to *find the bound on the time-varying transmission period when the feedback path is networked such that the NCS is still stable.* Previous work on this problem may be found in [5, 8]. However, those results might be too conservative to be of practical use:

Example 1 Consider the state-space plant model

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u, \\ y &= [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned} \tag{1}$$

A continuous-state feedback controller is $u = -Kx$, where $K = [3.75,\ 11.5]$ (closed-loop poles at $-1/2$ and $-3/4$). Using the theorem of [5], for $p = 1$ (only one network node, which is a non-networked sampled-data system), we obtain $\tau = 2.7 \times 10^{-4}$ s. By randomly selecting Q and solving for P, we can calculate τ using the corollary in [8]. In 200 trials, the maximum τ found was 4.5×10^{-4} s. However, the maximum stable constant sampling period for this feedback control system is 1.7 s.

In this paper, we take a different approach to derive the bound on the transmission period. We use a hybrid systems stability analysis technique. Then, we derive theorems for bounding the transmission period, based on a variety of different methods. We also provide a computational approach to numerically search for such a bound.

2 NCSs with Time-Varying Transmission Period

Let the original non-networked continuous system be represented by

$$\begin{aligned} \dot{z}(t) &= Az(t) + Bu(t), \\ y(t) &= Cz(t), \end{aligned} \tag{2}$$

where $z \in \boldsymbol{R}^n, u \in \boldsymbol{R}^m, y \in \boldsymbol{R}^p$, and $A \in \boldsymbol{R}^{n\times n}, B \in \boldsymbol{R}^{n\times m}, C \in \boldsymbol{R}^{p\times n}$. $z(t)$ is the state vector of the continuous system. We assume the open-loop system is unstable and can be stabilized by a full-state feedback gain controller

$$u(t) = -Kz(t), \tag{3}$$

where $K \in \boldsymbol{R}^{m\times n}$. Therefore, the closed-loop continuous system is given by

$$\dot{z}(t) = (A - BK)z(t). \tag{4}$$

Defining $\bar{A} = A - BK$, we have $\bar{A}$ is Hurwitz. Let $V(z(t)) = z^T(t)Pz(t)$ be the Lyapunov function of the closed-loop system that satisfies the Lyapunov equation

$$P\bar{A} + \bar{A}^T P = -Q, \tag{5}$$

where P and Q are positive definite symmetric matrices.

Now suppose the feedback control loop is closed through a communication network, and the full-state information is sent in one packet, as shown in Figure 1. The system equation of the networked system can be written as

$$\begin{array}{rcll} \dot{x}(t) & = & Ax(t) - BK\hat{x}(t), & t \in [t_k, t_{k+1}), \\ \hat{x}(t_k^+) & = & x(t_k), & k = 0, 1, \ldots, \end{array} \tag{6}$$

where $x(t) \in \boldsymbol{R}^n$ is the state vector of the networked system, $\hat{x}(t)$ is piecewise continuous and only changes value at t_k, and t_k, $k = 0, 1, \ldots$, is the sampling instant or transmission instant. The equation says that the control is updated at the instant t_k and kept constant until the next control update is received at time t_{k+1}.

Let h be the transmission period between successive transmissions. Define the transmission period at t_k as

$$h_k \triangleq t_{k+1} - t_k.$$

Our goal is to find an upper bound, h_{suff}, on h_k for $k = 0, 1, \ldots$ such that the networked system is still exponentially stable (a sufficient condition). Let h_{true} denote the true bound on h_k (a necessary and sufficient condition), and h_{nece} be a necessary condition. h_{schur} is the bound when constant transmission period is employed. We are finding some bound $h_{\text{suff}} \leq h_{\text{true}}$, therefore it is a sufficient condition.

We define the transmission error $e(t)$ as

$$e(t) \triangleq x(t) - x(t_k), \qquad t \in [t_k, t_{k+1}), \tag{7}$$

$e(t)$ is piecewise continuous and is reset to 0 at every transmission instant, i.e. $e(t_k) = 0$. With this definition, Equation (6) can be rewritten as

$$\begin{array}{rcl} \dot{x}(t) & = & Ax(t) - BKx(t_k), \\ & = & (A - BK)x(t) + BK(x(t) - x(t_k)), \\ & = & \bar{A}x(t) + BKe(t). \end{array} \tag{8}$$

3 Theoretical Results

In this section, several lemmas are derived, they are concerning with the bound on the transmission errors $e(t)$, the bound on the state error $x(t) - z(t)$, and the lower bound on the decrease of the Lyapunov function $V(z(t))$. To prove our theorems, we derive a few straightforward lemmas (see [7] for proofs) and apply a useful form of Bellman-Gronwall lemma (Lemma 3.3.8 in [3], hereafter B-G).

3.1 Useful Bounds

This lemma gives a bound on the transmission error $e(t)$ between successive transmissions at t_k and t_{k+1}.

Lemma 2 (Transmission Error Upper Bound) *The transmission error $e(t)$ defined in Equation (7) is bounded by*

$$\|e(t)\| \leq \frac{\|\bar{A}\|}{\|A\|}\left(e^{\|A\|t} - 1\right)\|x(t_k)\|, \qquad t \in [t_k, t_{k+1}) \tag{9}$$

between two successive transmissions.

Assumption 3 *It is reasonable to assume that there is a maximum transmission period $h_{\max}$, which means that the NCS will force a transmission of the plant state if the controller does not receive the information for $h_{\max}$ time. This can be done using the Controller Area Network (CAN), for instance, by making this message the highest priority. Assuming this, the bound on $\|e(t)\|$ is given by*

$$\|e(t)\| \leq \frac{\|\bar{A}\|}{\|A\|}\left(e^{\|A\|h_{\max}} - 1\right)\|x(t_k)\|, \qquad t \in [t_k, t_{k+1}).$$

This lemma gives a bound on how much the trajectory of the networked system is going to diverge from the original system starting from the same initial condition. A more general result has also been obtained in [1].

Lemma 4 (State Error Upper Bound) *Starting from the same initial condition $z(t_k) = x(t_k)$, the state error between the continuous system given by Equation (2) and the networked system given by Equation (8) is bounded by*

$$\|x(t) - z(t)\| \leq E\left(e^{\|\bar{A}\|(t-t_k)} - 1\right)\|x(t_k)\|, \qquad t \in [t_k, t_{k+1}),$$

where

$$E = \frac{\|BK\|\left(e^{\|A\|h_{\max}} - 1\right)}{\|A\|}.$$

This lemma gives a lower bound on the decrease of the Lyapunov function. In other words, starting from the initial condition, the Lyapunov function at least decreases this amount during the time interval.

Lemma 5 (Lower Bound on Decrease of Lyapunov Function) *Assume that $V(z(t)) = z^T P z$ is the Lyapunov function of the continuous system described by Equation (4) and satisfies Equation (5). Let $z(t_k)$ be the initial condition, the decrease of the Lyapunov function is lower bounded by*

$$\|V(z(t_k)) - V(z(t))\| \geq D\left(1 - e^{-2\|\bar{A}\|(t-t_k)}\right)\|z(t_k)\|^2$$

for $t > t_k, k = 0, 1, \ldots,$ where

$$D = \frac{\lambda_{\min}(Q)}{2\|\bar{A}\|}.$$

3.2 Stability of NCSs

Without loss of generality, assume that **0** is an equilibrium of the (continuous and NCS) systems, the systems start at $t_0 = 0$ with initial condition $x(0)$.

The following lemma guarantees the NCS described by Equation (6) is asymptotically stable. It is a consequence of state boundedness between sampling instants and Theorem 2.3 in [2]. See [4] for background definitions.

Lemma 6 (Stability of NCSs) *The NCS described by Equation (6) is uniformly asymptotically stable if there exists a continuous differentiable, locally positive definite function* $V : R^n \to R_+$ *and functions* α, β, γ *of class* K *such that for all* $x \in B_r$

$$\alpha(\|x\|) \leq V(x) \leq \beta(\|x\|), \tag{10}$$

and

$$\triangle V_k \triangleq V(x(t_{k+1})) - V(x(t_k)) \leq -\gamma(\|x(t_k)\|), \quad k = 0, 1, \ldots. \tag{11}$$

Proof. See [7]. □

Lemma 6 is only concerned with the Lyapunov function's decreasing at sampling instants; it does not require the Lyapunov function to be strictly decreasing over time, i.e. $\dot{V}(x(t)) < 0$, as required in Walsh *et al.*'s proof [5]. Figure 2 depicts a Lyapunov function that satisfies this constraint. We see that although $V(x(t))$ is increasing during some time intervals, it decreases at every sampling instant. Therefore it is a valid Lyapunov function to prove the stability of the NCS. Using this stability condition, we envision to derive less conservative bounds on the transmission period.

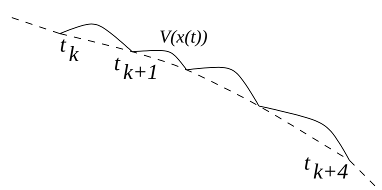

Figure 2: A valid Lyapunov function to prove NCS stability

4 Bound on Transmission Period

Now we derive the theorem for the upper bound on the transmission period.

Theorem 7 (Upper Bound on the Transmission Interval) *Let* $h_{\max}$ *be the maximum transmission period (defined by the NCS setup) and satisfy*

$$h_{\max} < \frac{1}{\|A\|} \ln\left(\frac{\lambda_{\min}(Q)\|A\|}{2\lambda_{\max}(P)\|BK\|\|\bar{A}\|} + 1\right). \tag{12}$$

Define the polynomial

$$p(x) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1 x + a_0, \tag{13}$$

where

$$
\begin{aligned}
a_4 &= \lambda_{\max}(P)(E^2 + 2E), \\
a_3 &= -2\lambda_{\max}(P)(E^2 + E), \\
a_2 &= \lambda_{\max}(P)E^2 - \frac{\lambda_{\min}(Q)}{2\|\bar{A}\|}, \\
a_1 &= 0, \\
a_0 &= \frac{\lambda_{\min}(Q)}{2\|\bar{A}\|}.
\end{aligned}
$$

1. *If $p(x)$ has real root(s) greater than 1, let $x^* > 1$ be the root that is closest to 1, the system is exponentially stable if*

$$h_{\text{suff}} = \min\left\{ h_{\max}, \frac{1}{\|\bar{A}\|} \ln x^* \right\},$$

2. *If $p(x)$ does not have real root greater than 1, then the system is exponentially stable if $h_{\text{suff}} = h_{\max}$.*

Proof. See [7]. □

Theorem 7 involves solving for the roots of a fourth-order polynomial, which can be easily done using MATLAB. The following example illustrates the usage of the theorem.

Example 8 Consider the same setup as in Example 1, the closed-loop system matrix is

$$\bar{A} = A - BK = \begin{bmatrix} 0 & 1.0 \\ 0.375 & -1.25 \end{bmatrix}.$$

h_{suff} decided by Theorem 7 depends on the selection of Q matrix in Equation (5). Our method to find h_{suff} is to randomly select a Q, gradually increase $h_{\max}$ in the range decided by Equation (12), and solve for h_{suff}. The maximum h_{suff} for the different $h_{\max}$'s for this particular Q is recorded. The process is repeated many times to find the maximum h_{suff} for different Q's. In 1000 trials, the maximum h_{suff} we have found is $h_{\text{suff}} = 0.0593$ s. This result is obtained by setting $h_{\max} = 0.0594$ s and selecting

$$Q = \begin{bmatrix} 6.5657 & 0.1151 \\ 0.1151 & 6.7410 \end{bmatrix}.$$

The real root of $p(x)$ that is closest to 1 in this setup is $x^* = 1.1014$. Therefore,

$$h_{\text{suff}} = \min\left\{ 0.0594, \frac{1}{\|\bar{A}\|} \ln 1.1014 \right\} = 0.0593 \text{ s}.$$

This says that the transmission period's always being less than 0.0593 s is sufficient to guarantee the exponential stability of this NCS. Comparing to the bound (4.5×10^{-4} s) we obtained in Example 1, this theorem is much less conservative.

5 Alternate Bound on Transmission Period

In our above derivation, we used a conservative bound to bound the exponential of a Hurwitz matrix, i.e. $\|e^{At}\| \leq e^{\|A\|t}$. In this section, we prove an alternate bound on the transmission period using a different bound on the exponential of a Hurwitz matrix. We only present the final result here and defer the proofs to [7], to keep our discussion smooth.

Theorem 9 (Alternate Bound on the Transmission Period) *Assuming $\bar{A} \in \boldsymbol{R}^{n\times n}$ has n distinct eigenvalues, let $h_{\max}$ be the maximum transmission period (defined by the NCS setup) and satisfies*

$$h_{\max} < \frac{1}{\|A\|} \ln\left(\frac{\lambda_{\min}(Q)\|A\|}{c^4\lambda_{\max}(P)\|BK\|\|\bar{A}\|} + 1\right), \tag{14}$$

define the polynomial

$$p(x) = a_{2\bar{r}}x^{2\bar{r}} + a_2x^2 + a_1x + a_0, \tag{15}$$

where

$$\begin{aligned} a_{2\bar{r}} &= \tilde{D}, \\ a_i &= 0, \quad i = 3, \ldots, (2\bar{r}-1) if \bar{r} > 1, \\ a_2 &= \lambda_{\max}(P)\tilde{E}^2 - \lambda_{\max}(P)c\tilde{E}, \\ a_1 &= \lambda_{\max}(P)c\tilde{E} - 2\lambda_{\max}(P)\tilde{E}^2, \\ a_0 &= \lambda_{\max}(P)\tilde{E}^2 - \tilde{D}. \end{aligned}$$

1. *If $p(x)$ has real root(s) less than 1, then let $x^* < 1$ be the root that is closest to 1, the system is exponentially stable if*

$$h_{\text{suff}} = \min\left\{h_{\max}, \frac{1}{\eta}\ln x^*\right\},$$

2. *If $p(x)$ does not have real root less than 1, then the system is exponentially stable if $h_{\text{suff}} = h_{\max}$.*

Proof. Follows the reasoning in the proof of Theorem 7. See [7]. □

6 Comparison of the Two Theorems

The two theorems (Theorem 7 and Theorem 9) we have derived are applicable to different kinds of plants. The difference between the two theorems are using two different bounds to bound the exponential of a matrix. In Theorem 7, we use $\left\|e^{\bar{A}t}\right\| \leq e^{\|\bar{A}\|t}$, while in Theorem 9 we use $\left\|e^{\bar{A}t}\right\| \leq ce^{\eta t}$. The first bound is conservative in the way that we bound a decreasing exponential of a Hurwitz matrix using an increasing bound. The second bound is less conservative because η can be set less than 0, therefore it can track the decrease of the exponential. However, when it is applied to derive Theorem 9, it does not necessarily lead to a less conservative result. The reason for this is: when t is close to zero, both $e^{\|A\|t}$ and $e^{\eta t}$ are close to 1, therefore the value of c becomes dominant in the bound. Hence, if c has a large value, the second bound will be conservative at the starting point, even though it is less conservative in the long run. This effect will be obvious in the following example.

k	20	30	40
Theorem 7	0.0272 s	0.0157 s	0.0110 s
Theorem 9	0.0513 s	0.0347 s	0.0265 s
h_{true}	0.1297 s	0.0732 s	0.0526 s

Table 1: Comparison of bounds for a scalar plant with different closed-loops

Example 10 Consider the open-loop scalar plant $\dot{x} = 15x + u$ with feedback gain $k = 20$. We can see that first bound is given by $\|e^{-5t}\| \leq e^{5t}$, while the second bound is given by $\|e^{-5t}\| \leq e^{-5t}$, with $c = 1$ and $\eta = -5$. Using Theorem 7 and the same searching method as in Example 8, the maximum h_{suff} we obtained is $h_{\text{suff}} = 0.0272$ s, when $h_{\max}$ is set at 0.0272 s. The real root of $p(x)$ in Theorem 7 that is closest to 1 is $x^* = 1.1527$. Therefore,

$$h_{\text{suff}} = \min\left\{0.0272, \frac{1}{5}\ln 1.1527\right\} = 0.0272 \text{ s}.$$

Meanwhile, based on Theorem 9 and the same searching strategy, the maximum h_{suff} we obtained is $h_{\text{suff}} = 0.0513$ s. This happens when $h_{\max}$ is set at 0.0513 s, and the real root of $p(x)$ in Theorem 9 that is closest to 1 is $x^* = 0.7534$. Therefore,

$$h_{\text{suff}} = \min\left\{0.0513, \frac{1}{-5}\ln 0.7534\right\} = 0.0513 \text{ s}.$$

We conclude from the above example that Theorem 9 gives less conservative results. However, this is not always the case, which is shown in the next example.

Example 11 Consider the state-space plant given in Example 8, the less conservative bound on $\left\|e^{\bar{A}t}\right\|$ is given by $\left\|e^{\bar{A}t}\right\| = 11.09\ e^{-0.5t}$, where c has a higher value. The calculation of $h_{\max}$ based on Equation (14) gives $h_{\max} < 1.04 \times 10^{-5}$ s due to the higher value of c. In fact, further calculation shows $h_{\text{suff}} = h_{\max}$.

Example 12 (Scalar Plants) For the scalar plant case, the analytic bound on the transmission period h_{true} can easily calculated by hand. Let the general scalar plant be represented by

$$\begin{aligned} \dot{x} &= ax(t) - k\hat{x}(t), \\ \hat{x}(t^+) &= x(t_k), \end{aligned} \tag{16}$$

after some simple algebra, we obtain

$$h_{\text{true}} = \frac{1}{a}\ln\frac{\frac{k}{a}+1}{\frac{k}{a}-1}. \tag{17}$$

Hence, if $h_k < h_{\text{true}}$, the networked scalar plant is stable.

With the true analytic bound in hand, we can immediately see how conservative our theorems would be for the scalar case. Consider an open-loop scalar plant $\dot{x} = 15x + u$, with different feedback gains $k = 20, 30, 40$. The bounds given by the analytic result, Theorem 7, and Theorem 9 are given in the following table. We can still see conservativeness in both theorems.

There is one more useful way to interpreting the results of Example 12. Results from the true bound, Theorem 7, and Theorem 9 all show that the faster plant requires a shorter transmission period (or faster transmission rate). These results are in agreement with our intuitive idea that more communication bandwidth should be allocated to faster plants.

7 Numerical Search for the Bound

A numerical search can also be conducted for finding the bound on the transmission period h. Based on Lemma 6, we can numerically search for the h_{suff} such that if $h_k \in [0, h_{\text{suff}}]$, then $V(x(t_k + h_k)) - V(x(t_k)) < 0$ for all k = 0, 1, ... Therefore, $h_k < h_{\text{suff}}$ for all $k = 0, 1, \ldots$ is a sufficient condition for the NCS represented by Equation (6) to be exponentially stable.

We know that the solution of Equation (6) at $t_k + h$ starting from t_k is given by $x(t_k + h) = \tilde{\Phi}(h_k)x(t_k)$, where $\tilde{\Phi}(h) = e^{Ah} - \int_0^h e^{As} ds BK$. Then

$$\begin{aligned} V(x(t_k + h)) - V(x(t_k)) &= x^T(t_k + h)Px(t_k + h) - x^T(t_k)Px(t_k), \\ &= x^T(t_k)\left[\tilde{\Phi}^T(h)P\tilde{\Phi}(h) - P\right]x(t_k). \end{aligned}$$

Defining $G(P, h) = \tilde{\Phi}^T(h)P\tilde{\Phi}(h) - P$, we want to find the bound on h, h_{suff}, such that for all $h \in [0, h_{\text{suff}}]$, $G(P, h)$ is negative definite. In summary, we have proved

Theorem 13 *Given $P = P^T > 0$ and $G(P, h)$ is negative definite for all $h \in [0, h_{\text{suff}}]$, the NCS in Equation (6) is exponentially stable if its transmission period $h_k \in [0, h_{\text{suff}}]$ for all $k = 0, 1, \ldots$.*

We also notice that when h_k is constant for all $k = 0, 1, \ldots$, the above equation reduces to the Lyapunov equation of discrete LTI systems. In this case, we only need to guarantee that $\tilde{\Phi}$ is Schur (i.e., all its eigenvalues have magnitude less than one) for the NCS to be stable. Let h_{schur} denote the bound on the constant h for $\tilde{\Phi}$ to be Schur.

In the case of time-varying transmission period, testing the Schur-ness of $\Phi(\tilde{h}_k)$ for all $k = 0, 1, \ldots$ is not sufficient to guarantee the NCS to be stable. The reason is that one can easily find an example of two Schur matrices whose multiplication is not Schur.

Example 14 Consider the two matrices given below

$$A = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ \alpha & 0 \end{pmatrix}.$$

A and B are always Schur, but AB is not Schur when $|\alpha| > 1$.

It is also possible to find example NCSs that are stable at constant sampling periods, yet unstable when the the sampling period is varying.

Example 15 Consider the continuous-time system described by Equation (2) whose A, B, K matrices are given by

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.6 \end{bmatrix}, \quad K = \begin{bmatrix} 1 & 6 \end{bmatrix}.$$

We see that $\bar{A} = A - BK$ is Hurwitz since $\lambda_i(\bar{A}) = -1.30 \pm 1.58j$, $i = 1,2$. For this system, h_{schur}, obtained by the exhaustive search described below, is 0.59 s (the search grid is 0.01). Therefore, the NCS is stable when constant transmission periods $h_1 = 0.18$ s or $h_2 = 0.54$ s is employed. This can be verified by testing the Schur-ness of $\tilde{\Phi}(h_1)$ and $\tilde{\Phi}(h_2)$. Both of them are Schur since

$$\begin{aligned} \left|\lambda_i(\tilde{\Phi}(h_1))\right| &= 0.7761, \quad i = 1,2; \\ \left|\lambda_i(\tilde{\Phi}(h_2))\right| &= 0.7083, \quad i = 1,2. \end{aligned}$$

However, when the transmission period is varying in a periodic pattern, say $h_1 \to h_2 \to h_1 \to h_2 \ldots$, the NCS is unstable. This is because $\tilde{\Phi}(h_1)\tilde{\Phi}(h_2)$ is not Schur, since

$$\max_{i=1,2}\left\{\left|\lambda_i\left(\tilde{\Phi}(h_1)\tilde{\Phi}(h_2)\right)\right|\right\} = 1.0049 > 1.$$

Our algorithm for exhaustive search tries find h_{suff} by increasing h along a tight grid from 0 and testing the eigenvalues of $G(P,h)$. Whenever the search yields eigenvalues with non-negative real parts, the search stops, and the h_{suff} is the h from the last step. By randomly selecting Q and solving for P based on the Equation (5), the search is carried out many times and we record the maximum h_{suff} we ever find. A similar search can also be conducted for the bound on the constant h, h_{schur}, by testing the Schur-ness of the matrix $\tilde{\Phi}(h)$.

Example 16 (Exhaustive Search) Let us still consider the system as in Example 8, setting the search grid as 0.0001 s and carrying out the exhaustive search by randomly selecting Q, the maximum h_{suff} we have obtained is $h_{\text{suff}} = 1.7294$ s. The Q and P in Equation (5) that leads to this result are

$$Q = \begin{bmatrix} 3.5109 & 7.2526 \\ 7.2526 & 19.7530 \end{bmatrix}, \qquad P = \begin{bmatrix} 2.9663 & 4.6812 \\ 4.6812 & 11.6462 \end{bmatrix}.$$

To hit this maximum h_{suff}, the search took 3.7 hours on a dual-CPU Pentium-III 450MHz computer running Linux operating system. This Q was the 63rd Q selected. Note that the MATLAB program was not optimized for time. The exhaustive search for the h_{schur} of this example gives the same result, i.e. $h_{\text{schur}} = 1.7294$ s. So the bound is tight.

References

[1] M.S. Branicky, Continuity of ODE solutions, *Appl. Math. Letts.*, **7(5)**:57–60, 1994.

[2] M.S. Branicky, Multiple Lyapunov functions and other analysis tools for switched and hybrid systems, *IEEE Trans. Automatic Control*, **43(4)**:475–482, April 1998.

[3] P.A. Ioannou and J. Sun, *Robust Adaptive Control*, Prentice Hall, 1996.

[4] M. Vidyasagar, *Nonlinear System Analysis*, Prentice Hall, 1993. 2nd ed.

[5] G.C. Walsh, H. Ye, and L. Bushnell, Stability analysis of networked control systems, in *Proc. American Control Conf.*, pp. 2876–2880, San Diego, June 1999.

[6] B. Wittenmark, J. Nilsson, and M. Törngren, Timing problems in real-time control systems, in *Proc. American Control Conf.*, pp. 2000–2004, Seattle, June 1995.

[7] W. Zhang, *Stability Analysis of Networked Control Systems*, Ph.D. Dissertation, Electrical Engineering and Computer Science Dept., Case Western Reserve Univ., May 2001. `http://dora.cwru.edu/msb/pubs/wxzPHD.pdf`

[8] W. Zhang, M.S. Branicky, and S.M. Phillips, Stability of networked control systems, *IEEE Control Systems Magazine*, **21(1)**:84–99, February 2001.