Nice Reachability for Planar Bilinear Control Systems with Applications to Planar Linear Switched Systems

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Abstract

We consider planar bilinear control systems with measurable controls. We show that any point in the reachable set can be reached by a "nice" control. Specifically, a control that is a concatenation of a bang arc with either (1) a bang-bang control that is periodic after the third switch; or (2) a piecewise constant control with no more than two discontinuities.

Under the additional assumption that the bilinear system is positive (or invariant for any proper cone), we show that the reachable set is spanned by a concatenation of a bang arc with either (1) a bang-bang control with no more than two discontinuities; or (2) a piecewise constant control with no more than two discontinuities. In particular, any point in the reachable set can be reached using a piecewise-constant control with no more than three discontinuities.

Several known results on the stability of planar linear switched systems under arbitrary switching follow as corollaries of our main result. We demonstrate this using one example.

Index Terms

Switched systems; optimal control; maximum principle; stability under arbitrary switching; Lie algebra; Lie brackets; positive linear systems; Metzler matrices.

I. INTRODUCTION

Consider the planar bilinear control system:

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + u(t)B\boldsymbol{x}(t), \qquad u \in \mathcal{U}, \tag{1}$$

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where $\boldsymbol{x}(\cdot) : \mathbb{R}_+ \to \mathbb{R}^2$, $A, B \in \mathbb{R}^{2\times 2}$, and $\mathcal{U}$ is the set of measurable functions taking values in $[0,1]$. For $u \in \mathcal{U}$, and $T \geq 0$, we use $\boldsymbol{x}(T; u, \boldsymbol{x}_0)$ to denote the solution at time T of (1) for the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$.

Our main result is a nice reachability type result (see, e.g., [27], [26]) for (1). In other words, if $R(T; \mathcal{V}, \boldsymbol{x}_0) := \{\boldsymbol{x}(T; v, \boldsymbol{x}_0) : v \in \mathcal{V}\}$ denotes the reachable set for some subset of controls $\mathcal{V} \subseteq \mathcal{U}$, then we show that $R(T; \mathcal{U}, \boldsymbol{x}_0) = R(T; \mathcal{W}, \boldsymbol{x}_0)$, where $\mathcal{W} \subset \mathcal{U}$ is some subset of "nice" controls. (Here, "nice" may mean, for example, the set of continuous controls or the set of piecewise-constant controls, etc.)

This implies that for $\boldsymbol{x}(0) = \boldsymbol{x}_0$, any point that can be reached at time T using a control $u \in \mathcal{U}$ can also be reached, at the same time, using some "nice" control $w \in \mathcal{W}$. Nice reachability results have many theoretical and practical applications. Indeed, any point-to-point control problem (e.g., motion planning, finding optimal controls) over the set $\mathcal{U}$ can be reduced to the problem of finding a suitable control from the set of "nice" controls $\mathcal{W}$.

Some related papers on the analysis of planar control systems include the following. The remarkable set of papers by Sussmann [28], [29], [30] provides nice reachability results for *nonlinear* control systems in the plane (see also [3, Chapter 3]). Since we make a stronger assumption, namely, that the system is bilinear, we are able to derive a stronger result. In particular, Sussmann's results hold in general only for a sufficiently small final time $T \geq 0$, whereas our results hold for all $T \geq 0$. Another important paper is [21], which provides a nice analysis of the structure of optimal bang-bang controls for planar bilinear systems. This paper, however, ignores the possibility of optimal controls that are not bang-bang. Also, our analysis of the bang-bang case (see Section IV-A below) is much simpler than the one given in [21].

Our results have ramifications for switched and hybrid systems [4], [5], [14], [6], [13], particularly for the stability analysis of linear switched systems under arbitrary switching laws.

The remainder of this paper is organized as follows: the main results are stated in the next section. The proofs of these results, given in Section V, are based on a variational approach. This approach is briefly reviewed in Section III, and applied to the analysis of time-optimal controls of the so-called auxiliary system in Section IV.

II. MAIN RESULTS

We introduce some notation. Given two controls $u_1 : [0, T_1] \to [0, 1]$ and $u_2 : [0, T_2] \to [0, 1]$, we use $u_2 * u_1$ to denote their concatenation, that is,

$$(u_2 * u_1)(t) := \begin{cases} u_1(t), & t \in [0, T_1), \\ u_2(t - T_1), & t \in [T_1, T_1 + T_2]. \end{cases}$$

The corresponding trajectory is obtained by first following u_1 and then u_2. For $\mathcal{U}_1, \mathcal{U}_2 \subseteq \mathcal{U}$, we use $\mathcal{U}_2 * \mathcal{U}_1$ to denote the set of all concatenations $u_2 * u_1$ where, for $i = 1, 2$, either $u_i \in \mathcal{U}_i$ or u_i is trivial (that is, the domain of u_i includes a single point). Hence, $\mathcal{U}_2 * \mathcal{U}_1$ essentially contains both $\mathcal{U}_1$ and $\mathcal{U}_2$ themselves. For example, if $\mathcal{BB}_k \subset \mathcal{U}$ denotes the set of bang-bang controls with no more than k discontinuities, then $(\mathcal{BB}_1 * \mathcal{BB}_2) = \mathcal{BB}_4$ (note that the concatenation may introduce an additional discontinuity).

Consider a bang-bang control u with switching times $T_1 < T_2 < T_3 < \dots$, that is, $u(t) = v$ for $t \in [0, T_1)$, $u(t) = 1 - v$ for $t \in [T_1, T_2)$, and so on with $v \in \{0, 1\}$. Denote $T_{ij} := T_i - T_j$. We say that u is periodic after three switches if $T_{21} = T_{43} = T_{65} = \dots$ and $T_{32} = T_{54} = T_{76} = \dots$. Let $\mathcal{BP} \subset \mathcal{U}$ denote the set of such controls, and let $\mathcal{PC}_k \subset \mathcal{U}$ denote the set of piecewise constant functions with no more than k discontinuities. We can now state our main result.

Theorem 1 *Denote $\mathcal{W} := (\mathcal{BB}_0 * \mathcal{BP}) \cup (\mathcal{BB}_0 * \mathcal{PC}_2)$. The reachable set of* (1) *satisfies*

$$R(T; \mathcal{U}, \boldsymbol{x}_0) = R(T; \mathcal{W}, \boldsymbol{x}_0), \qquad \text{for all } \boldsymbol{x}_0 \in \mathbb{R}^2 \text{ and all } T \geq 0. \tag{2}$$

Theorem 1 can be strengthened if the control system admits an invariant cone. A set $C \subseteq \mathbb{R}^n$ is called a *convex cone* if $\boldsymbol{p}, \boldsymbol{q} \in C$ implies that $k_1 \boldsymbol{p} + k_2 \boldsymbol{q} \in C$ for all $k_1, k_2 \geq 0$. The cone is said to be: *solid* if its interior is non-empty; *pointed* if $C \cap (-C) = \{\boldsymbol{0}\}$; *proper* if it is both solid and pointed.

Theorem 2 *Consider the control system* (1). *Suppose that there exists a proper cone $C \subset \mathbb{R}^2$ that is an invariant set for both $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ and $\dot{\boldsymbol{x}} = (A + B)\boldsymbol{x}$. Let $\mathcal{V} := \mathcal{BB}_3 \cup (\mathcal{BB}_0 * \mathcal{PC}_2)$. Then*

$$R(T; \mathcal{U}, \boldsymbol{x}_0) = R(T; \mathcal{V}, \boldsymbol{x}_0), \qquad \text{for all } \boldsymbol{x}_0 \in \mathbb{R}^2 \text{ and all } T \geq 0.$$

Note that since $\mathcal{V} \subset \mathcal{PC}_3$, this implies that any point-to-point control problem is reduced to the problem of determining a (small) set of parameters: the three switching times and the four constant control values between the switches.

We say that (1) is *globally asymptotically stable* (GAS) if there exists a class $\mathcal{KL}$ function[1] β such that for any initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and *any* control $u \in \mathcal{U}$: $|\boldsymbol{x}(t; u, \boldsymbol{x}_0)| \leq \beta(|\boldsymbol{x}_0|, t)$, for all $t \geq 0$. The difficulty in analyzing GAS stems from the fact that the set of solutions $\{\boldsymbol{x}(\cdot; u, \boldsymbol{x}_0) : u \in \mathcal{U}\}$ is huge. One possible application of nice reachability results is in establishing GAS results. The next result demonstrates this. For $c \in [0, 1]$, we use $\boldsymbol{x}(t; c, \boldsymbol{x}_0)$ to denote the solution of (1) for the constant control $u(t) \equiv c$.

Proposition 1 *Let $D \subseteq [0, 1]$ be some subset of control values. Suppose that there exists a $\bar{\beta} \in \mathcal{KL}$ such that*

$$|\boldsymbol{x}(t; c, \boldsymbol{x}_0)| \leq \bar{\beta}(|\boldsymbol{x}_0|, t), \qquad \text{for any } c \in D. \tag{3}$$

Suppose also that $R(T; \mathcal{U}, \boldsymbol{x}_0) = R(T; \mathcal{PC}_k(D), \boldsymbol{x}_0)$ for all $\boldsymbol{x}_0 \in \mathbb{R}^n$ and all $T \geq 0$, where k is an integer that does not depend on T and $\boldsymbol{x}_0$, and $\mathcal{PC}_k(D)$ is the set of piecewise constant controls with no more than k discontinuities taking values in D. Then the control system (1) *is GAS.*

Proof. A straightforward generalization of a construction from [15] (see also the proof of Proposition 1 in [24]) allows one to conclude that there exists a class $\mathcal{KL}$ function β such that for any control in $\mathcal{PC}_k(D)$, $|\boldsymbol{x}(T)| \leq \beta(|\boldsymbol{x}_0|, T)$. The fact that k is fixed independent of $\boldsymbol{x}_0$ and T is crucial to the argument. $\square$

The intuition behind this result can be explained as follows: (3) implies that instability for controls taking values in D can only be obtained by a control u that includes *repeated switchings*. However, since u can be replaced by a control law $v \in \mathcal{PC}_k(D)$, no control can induce instability.

Note that in the bilinear case, $\boldsymbol{x}(t; c, \boldsymbol{x}_0) = \exp((A + cB)t)\boldsymbol{x}_0$, so if D is compact, then (3) is equivalent to the requirement that $A + cB$ is Hurwitz for all $c \in D$. In particular, if $D = \{0, 1\}$ (so $\mathcal{PC}_k(D) = \mathcal{BB}_k$) then (3) is equivalent to the requirement that A and $A + B$ are Hurwitz.

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[1] Recall that a function $\alpha : [0, \infty) \to [0, \infty)$ is said to be of *class* $\mathcal{K}$ if it is continuous, strictly increasing, and $\alpha(0) = 0$. A function $\beta : [0, \infty) \times [0, \infty) \to [0, \infty)$ is said to be of *class* $\mathcal{KL}$ if $\beta(\cdot, t)$ is of class $\mathcal{K}$ for each fixed $t \geq 0$ and $\beta(s, t)$ decreases to 0 as $t \to \infty$ for each fixed $s \geq 0$.

A. Applications to Stability Analysis of Planar Linear Switched Systems

The GAS of (1) is closely related to the stability analysis of linear switched systems under arbitrary switching laws [14], [6], [13]. Consider the planar linear switched system

$$\dot{\boldsymbol{x}}(t) = A_{\sigma(t)}\boldsymbol{x}(t), \tag{4}$$

where $\boldsymbol{x}(\cdot) : \mathbb{R}_+ \to \mathbb{R}^2$, $\sigma(\cdot) : \mathbb{R}_+ \to \{0, 1\}$ is the *switching signal*, $A_0 = A$, and $A_1 = A + B$. We say that (4) is *globally uniformly asymptotically stable* (GUAS) if there exists a class $\mathcal{KL}$ function β such that for any initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and *any* switching law σ the corresponding solution of (4) satisfies: $|\boldsymbol{x}(t)| \leq \beta(|\boldsymbol{x}_0|, t)$, for all $t \geq 0$.

Remark 1 By definition, the set of solutions of (4) is contained in the set of solutions of (1). In particular, GAS of (1) immediately implies GUAS of (4).

Recently, the problem of establishing GUAS of (4) has attracted considerable interest [13]. It is not difficult to verify that several known results on GUAS of planar linear switched systems follow as corollaries of our main results.

To demonstrate this, consider the GUAS problem for *positive switched systems*. Recall that a linear system $\dot{\boldsymbol{x}} = A\boldsymbol{x}$, with $A \in \mathbb{R}^{n\times n}$, is called *positive* if $\mathbb{R}^n_+ := \{\boldsymbol{x} | x_i \geq 0, i = 1, \dots n\}$ is an invariant set of the dynamics, that is, if $\boldsymbol{x}(0) \in \mathbb{R}^n_+$ implies that $\boldsymbol{x}(t) \in \mathbb{R}^n_+$ for all $t \geq 0$. A necessary and sufficient condition for this is that A is a *Metzler matrix*, that is, all the non-diagonal elements of A are non-negative.

If both A_0 and A_1 are Metzler and $\boldsymbol{x}_0 \in \mathbb{R}^n_+$, then we refer to (4) as a *positive switched linear system*. Mason and Shorten [20], and independently David Angeli, conjectured that if every matrix in the convex hull of A_0 and A_1 is Hurwitz and Metzler then the switched system (4) is GUAS (for any order n). Recently, advancing the arguments in [9], Gurvits et al. [10] were able to prove that this conjecture is in general false. However, they were also able to show that it does hold when $n = 2$ (even when the number of matrices is arbitrary). Their proof for two matrices in the planar case is based on showing that the system admits a common quadratic Lyapunov function, that is, a function $V(\boldsymbol{x}) = \boldsymbol{x}^T P \boldsymbol{x}$, with $P > 0$, such that $PA_i + A_i^T P < 0$, $i = 0, 1$. (For more on the analysis of switched systems using quadratic Lyapunov functions, see [4], [25].)

Theorem 2 can be viewed as a generalization of this two-matrix planar result of Gurvits et al. [10]. Indeed, if A_0 and A_1 are Metzler then Theorem 2 holds for $C = \mathbb{R}^2_+$. The next result then follows immediately from Proposition 1.

Corollary 1 *Consider the control system* (1) *with* $A, A+B \in \mathbb{R}^{2\times 2}$ *Metzler. If* $A+cB$ *is Hurwitz for all* $c \in [0,1]$*, then the control system is GAS.*

By Remark 1, this implies of course that the switched system (4) is GUAS.

The remainder of this paper is devoted to the proof of Theorems 1 and 2.

III. PRELIMINARIES: THE VARIATIONAL APPROACH

E. S. Pyatnitsky [22], [23] was the first to suggest a variational approach for addressing the GAS problem for bilinear control systems. The basic idea was to derive a characterization of the "most destabilizing" of all possible control laws using a suitable optimal control problem. In this context, the variational approach has several advantages. First, it allows the application of sophisticated and powerful tools, such as the first- and higher-orders maximum principles (MPs) to stability analysis. Second, many of the results can be generalized to nonlinear control systems. Third, it allows the derivation of not only stability results, but more general nice-reachability-type results.

The variational approach was used to derive the most general stability results currently available for both (1) linear switched systems of order $n = 2$ [23], [18], [12], and $n = 3$ [2], [19]; and (2) nonlinear switched systems with a nilpotent Lie algebra [24].

We note in passing that the variational approach may also be important in other questions relating to switched systems, e.g., the synthesis of optimal switching laws [7], [5]. Recently, this approach was also used to address the problem of computing the root-mean-square gain of linear switched systems [17]. More information can be found in the recent survey paper [16], and the references therein. For the sake of completeness, we briefly review the components of the variational approach that are needed for our purposes.

A. The Auxiliary System and Nice Reachability

Consider the control system (1). Define $\boldsymbol{y}(t; u, \boldsymbol{x}_0) := \exp(-At)\boldsymbol{x}(t; u, \boldsymbol{x}_0)$. Note that $\boldsymbol{y}(0; u, \boldsymbol{x}_0) = \boldsymbol{x}_0$. For the sake of brevity, we will sometimes use the shorthand notation $\boldsymbol{y}(t)$ for $\boldsymbol{y}(t; u, \boldsymbol{x}_0)$.

Differentiating $\boldsymbol{y}(t)$ yields

$$\dot{\boldsymbol{y}}(t) = u(t)S(t)\boldsymbol{y}(t), \qquad S(t) := \exp(-At)B\exp(At). \tag{5}$$

Note that this is a drift-less, time-varying control system. Following [11], we refer to (5) as the *auxiliary system*. Clearly, we can rewrite (5) in the form

$$\boldsymbol{y}(T) = \exp\left(\int_0^T u(t)S(t)dt\right)\boldsymbol{y}(0),$$

so for any $c \in \mathbb{R}$:

$$c\boldsymbol{y}(T) = \exp\left(\int_0^T u(t)S(t)dt\right)(c\boldsymbol{y}(0)).$$

This implies that the system is *homogeneous*, i.e. if a control u steers some initial condition $\boldsymbol{y}(0)$ to the final condition $\boldsymbol{y}(T)$ at time T, then this control also steers $c\boldsymbol{y}(0)$ to the final condition $c\boldsymbol{y}(T)$ at time T.

Proposition 2 *Fix arbitrary $T \geq 0$, $\boldsymbol{q} \in \mathbb{R}^2$, and $u \in \mathcal{U}$. Denote $\boldsymbol{p} := \boldsymbol{x}(T; u, \boldsymbol{q})$, and $\boldsymbol{p}' := \exp(-AT)\boldsymbol{p}$, so u steers* (1) *[*(5)*] from $\boldsymbol{q}$ to $\boldsymbol{p}$ [$\boldsymbol{p}'$] at time T. Let $u^* \in \mathcal{U}$ be a control that steers* (5) *from $\boldsymbol{q}$ to $\boldsymbol{p}'$ in* minimal time,[2] *that is, $\boldsymbol{y}(T'; u^*, \boldsymbol{q}) = \boldsymbol{p}'$ for some minimal time $T' \in [0, T]$. Define a control $w \in \mathcal{U}$ by:*

$$w(t) = \begin{cases} u^*(t), & t \in [0, T'), \\ 0, & t \in [T', T]. \end{cases} \tag{6}$$

Then $\boldsymbol{x}(T; w, \boldsymbol{q}) = \boldsymbol{p}$.

Proof. By (5) and (6): $\boldsymbol{y}(T; w, \boldsymbol{q}) = \boldsymbol{y}(T', u^*, \boldsymbol{q}) = \boldsymbol{p}'$, so $\boldsymbol{x}(T; w, \boldsymbol{q}) = \exp(AT)\boldsymbol{p}' = \boldsymbol{p}$. □

Proposition 2 implies that any control can be replaced by a control that is a concatenation of a time-optimal control for the auxiliary system and a bang arc. A similar result holds also for the case of nonlinear control systems [26].

Given (5), the time-optimal control u^* can be characterized using the MP.

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[2] The definition of the set of admissible controls $\mathcal{U}$ implies that an optimal control $u^* \in \mathcal{U}$ indeed exists [8].

Theorem 3 *Suppose that* u^* *is a time-optimal control for* (5) *and denote* $\boldsymbol{y}^*(t) := \boldsymbol{y}(t; u^*, \boldsymbol{x}_0)$*. There exists* $\boldsymbol{\eta} : [0, T'] \to \mathbb{R}^n$*, satisfying*

$$\dot{\boldsymbol{\eta}}(t) = -u^*(t) S^T(t) \boldsymbol{\eta}(t), \tag{7}$$

such that

$$u^*(t) = \begin{cases} 0, & m(t) > 0, \\ 1, & m(t) < 0, \end{cases} \tag{8}$$

where

$$m(t) := \boldsymbol{\eta}^T(t) S(t) \boldsymbol{y}^*(t). \tag{9}$$

Furthermore, let L_0 *denote the linear span of the set of all Lie brackets containing at least one* B *term:* $L_0(\boldsymbol{y}) = \operatorname{span}(\{B\boldsymbol{y}, [A, B]\boldsymbol{y}, [A, [A, B]]\boldsymbol{y}, [B, [A, B]]\boldsymbol{y}, \dots\})$*. Then the restriction of* $\boldsymbol{\eta}(t)$ *on* $L_0(\boldsymbol{y}^*(t))$ *does not vanish for all* $t \in [0, T']$*.*

For a proof of the last assertion, see [26].

If the set $Z := \{t \in [0, T'] : m(t) = 0\}$ contains only isolated points, then (8) implies that $u^*(t)$ is piecewise constant and satisfies $u^*(t) \in \{0, 1\}$ for almost all t. Such a control is called *bang-bang*. If, on the other hand, $m(t) \equiv 0$ on an interval of time, then the corresponding control is called *singular* [1].

Differentiating the absolutely continuous function m, and using (5) and (7) yields

$$\dot{m}(t) = \boldsymbol{\eta}^T(t) \exp(-At)(\operatorname{ad}_A B) \exp(At) \boldsymbol{y}^*(t), \tag{10}$$

where $\operatorname{ad}_A B := BA - AB$ is the Lie commutator of the matrices A and B. Note that this implies that $\dot{m}$ is also an absolutely continuous function.

One important application of the auxiliary system is in deriving nice-reachability results [26]. The next example demonstrates this.

Example 1 Consider the case where A and B commute, i.e. $AB = BA$. In this case, (10) yields $\dot{m}(t) = 0$, so $m(t) \equiv c$. Since $m(t) = \boldsymbol{\eta}^T(t) B \boldsymbol{y}^*(t)$ and $L_0(\boldsymbol{y}^*) = \operatorname{span}\{B\boldsymbol{y}^*\}$, it follows from Theorem 3 that $c \neq 0$, so (8) implies that either $u^*(t) \equiv 0$ or $u^*(t) \equiv 1$. The control w

defined in (6) then satisfies $w \in \mathcal{BB}_1$, so

$$R(T;\mathcal{U},\boldsymbol{x}_0) = R(T;\mathcal{BB}_1,\boldsymbol{x}_0), \qquad \text{for all } T \text{ and all } \boldsymbol{x}_0.$$

Suppose, in addition, that A and $A{+}B$ are Hurwitz. Then there exist $c_i > 0$ such that $|\exp(At)\boldsymbol{x}_0| \leq c_1|\boldsymbol{x}_0|\exp(-c_2 t)$ and $|\exp((A+B)t)\boldsymbol{x}_0| \leq c_3|\boldsymbol{x}_0|\exp(-c_4 t)$. In other words, (3) holds for $D = \{0,1\}$, and Proposition 1 implies that (1) is GAS. Thus, the stability result follows as a corollary of the more general nice-reachability result. □

IV. ANALYSIS OF TIME-OPTIMAL CONTROLS

In this section, we analyze time-optimal controls for the auxiliary system (5). For two vectors $\boldsymbol{p},\boldsymbol{q} \in \mathbb{R}^2$, we use the notation $\boldsymbol{p} \propto \boldsymbol{q}$ to denote that $\boldsymbol{p} = c\boldsymbol{q}$, for some $c \neq 0$. Denote $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. It is straightforward to verify that if two vectors $\boldsymbol{p},\boldsymbol{q} \in \mathbb{R}^2 \setminus \{\boldsymbol{0}\}$ satisfy $\boldsymbol{p}^T\boldsymbol{q} = 0$, then $\boldsymbol{p} \propto J\boldsymbol{q}$.

A. The bang-bang case

Proposition 3 *Consider the system* (5) *with* $A, B \in \mathbb{R}^{2\times 2}$, *and* $\boldsymbol{y}(0) \in \mathbb{R}^2 \setminus \{\boldsymbol{0}\}$. *Suppose that the set* Z *contains only isolated points. Let* u^* *be a time-optimal bang-bang control with at least two consecutive switches:* $t_1 < t_2$. *Denote the corresponding trajectory by* $\boldsymbol{y}^*$, *and let* $\boldsymbol{x}^*(t) := \exp(At)\boldsymbol{y}^*(t)$. *Then*

$$\boldsymbol{x}^*(t_2) \propto (\mathrm{ad}_A\, B)B\boldsymbol{x}^*(t_1). \tag{11}$$

Proof. Considering Theorem 3, denote $\boldsymbol{\lambda}(t) := \exp(-A^T t)\boldsymbol{\eta}(t)$. Then $\dot{\boldsymbol{\lambda}} = -(A + u^* B)^T\boldsymbol{\lambda}$, and $m(t) = \boldsymbol{\lambda}^T(t)B\boldsymbol{x}^*(t)$. Since t_1 is a switching point, $\boldsymbol{\lambda}^T(t_1)B\boldsymbol{x}^*(t_1) = 0$, so

$$\boldsymbol{\lambda}(t_1) \propto JB\boldsymbol{x}^*(t_1). \tag{12}$$

Assume first that $m(t) > 0$ for $t \in I := (t_1, t_2)$, so $u^*(t) = 0$, $\dot{\boldsymbol{x}} = A\boldsymbol{x}$, and $\dot{\boldsymbol{\lambda}} = -A^T\boldsymbol{\lambda}$ for all $t \in I$. Fix an arbitrary $t \in I$, and let $s := t - t_1$. Then

$$\begin{aligned} m(t) &= \boldsymbol{\lambda}^T(t)B\boldsymbol{x}^*(t) \\ &= \boldsymbol{\lambda}^T(t_1)\exp(-As)B\exp(As)\boldsymbol{x}^*(t_1) \\ &= \boldsymbol{\lambda}^T(t_1)(\exp(-As)B\exp(As) - B)\boldsymbol{x}^*(t_1), \end{aligned}$$

where the last step follows from the fact that $m(t_1) = 0$. This can be written as

$$m(t) = \boldsymbol{\lambda}^T(t_1)(\exp(-As)B - B\exp(-As))\boldsymbol{x}^*(t). \tag{13}$$

Since $A \in \mathbb{R}^{2\times 2}$, the Cayley-Hamilton theorem asserts that there exist $a_0, a_1 \in \mathbb{R}$ such that

$$A^2 = a_1 A + a_0 I. \tag{14}$$

The expansion $\exp(At) = I + At/1! + (At)^2/2! + \cdots$ implies that there exist analytic functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$\exp(At) = f(t)I + g(t)A, \qquad \text{for all } t. \tag{15}$$

Differentiating (15) yields $A(fI+gA) = \dot{f}I+\dot{g}A$, and using (14) yields $\dot{f} = a_0 g$, and $\dot{g} = a_1 g + f$. It follows from (15) that (one possible choice for) the initial conditions of this set of differential equations is $f(0) = 1$ and $g(0) = 0$. Solving for g yields

$$g(t) = \begin{cases} \exp(a_1 t/2)\sinh(\sqrt{p}t)/\sqrt{p}, & p > 0, \\ \exp(a_1 t/2)t, & p = 0, \\ \exp(a_1 t/2)\sin(\sqrt{-p}t)/\sqrt{-p}, & p < 0, \end{cases} \tag{16}$$

where $p := a_0 + a_1^2/4$. Note that (16) implies that for any $p \geq 0$: $g(t) \neq 0$ for all $t \neq 0$.

Substituting (15) in (13) yields

$$m(t) = -g(-s)\boldsymbol{\lambda}^T(t_1)(\operatorname{ad}_A B)\boldsymbol{x}^*(t). \tag{17}$$

Denote $t_{21} := t_2 - t_1$. Then at the switching point t_2 this yields

$$0 = g(-t_{21})\boldsymbol{\lambda}^T(t_1)(\operatorname{ad}_A B)\boldsymbol{x}^*(t_2). \tag{18}$$

Suppose that $g(-t_{21}) = 0$. Since $t_{21} > 0$, (16) yields $p < 0$ and $\sin(-\sqrt{-p}t_{21}) = 0$. Using (16) again yields $g(t_{21}) = 0$, so $\boldsymbol{x}^*(t_2) = \exp(At_{21})\boldsymbol{x}^*(t_1) = (f(t_{21})I + g(t_{21})A)\boldsymbol{x}^*(t_1) = f(t_{21})\boldsymbol{x}^*(t_1)$. Note that since $\boldsymbol{y}(0) = \boldsymbol{x}(0) \neq \mathbf{0}$, we have $\boldsymbol{x}^*(t) \neq \mathbf{0}$ for all $t \geq 0$, so $f(t_{21}) \neq 0$. It then follows from the homogeneity of the system that $u^*(t) = 0$ for all $t \geq 0$. This contradicts our assumption that there exist at least two switching points.

Thus, $g(-t_{21}) \neq 0$, and (18) and (12) yield $\boldsymbol{x}^*(t_2) \propto J(\operatorname{ad}_A B)^T\boldsymbol{\lambda}(t_1) \propto J(\operatorname{ad}_A B)^T JB\boldsymbol{x}^*(t_1)$. A direct calculation shows that for all $A, B \in \mathbb{R}^{2\times 2}$, $J(\operatorname{ad}_A B)^T JB = (\operatorname{ad}_A B)B$, so (11) indeed holds.

Now suppose that $m(t) < 0$ for $t \in I$. Then $u^*(t) = 1$ for $t \in I$, and the dynamics is $\dot{\boldsymbol{x}}^* = (A + B)\boldsymbol{x}^*$ and $\dot{\boldsymbol{\lambda}} = -(A + B)^T\boldsymbol{\lambda}$ rather than $\dot{\boldsymbol{x}}^* = A\boldsymbol{x}^*$ and $\dot{\boldsymbol{\lambda}} = -A^T\boldsymbol{\lambda}$. Substituting $A + B$ for A in (11) yields $\boldsymbol{x}^*(t_2) \propto (\operatorname{ad}_{A+B} B)B\boldsymbol{x}^*(t_1) = (\operatorname{ad}_A B)B\boldsymbol{x}^*(t_1)$. This completes the proof of Proposition 3. □

Proposition 4 *Consider the system* (1) *with* $A, B \in \mathbb{R}^{2\times 2}$*, and* $\boldsymbol{y}(0) \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$*. Suppose that the set* Z *contains only isolated points. Let* u^* *be a time-optimal bang-bang control with more than two switches, and let* $t_1 < t_2 < t_3$ *denote three consecutive switching points. Denote the corresponding trajectory by* $\boldsymbol{y}^*$*, and let* $\boldsymbol{x}^*(t) := \exp(At)\boldsymbol{y}^*(t)$*. Then*

$$\boldsymbol{x}^*(t_3) \propto \boldsymbol{x}^*(t_1).$$

Remark 2 Note that it follows from this and the homogeneity of the system that $u^* \in \mathcal{BP}$. □

Proof. We already know that

$$\boldsymbol{x}^*(t_2) \propto (\operatorname{ad}_A B)B\boldsymbol{x}^*(t_1). \tag{19}$$

Consider the interval $t \in (t_2, t_3)$. The dynamics on this interval is identical to that in the interval (t_1, t_2) with one difference: A is replaced by $A + B$. It therefore follows from (19) that $\boldsymbol{x}^*(t_3) \propto (\operatorname{ad}_{A+B} B)B\boldsymbol{x}^*(t_2) = (\operatorname{ad}_A B)B\boldsymbol{x}^*(t_2)$. Combining this with (19) yields $\boldsymbol{x}^*(t_3) \propto ((\operatorname{ad}_A B)B)^2\boldsymbol{x}^*(t_1)$. A direct calculation shows that for all $A, B \in \mathbb{R}^{2\times 2}$, $((\operatorname{ad}_A B)B)^2 = p(A, B)I$, with $p(A, B) = -\det((\operatorname{ad}_A B)B)$, and this completes the proof. □

B. The singular case

Proposition 5 *Fix an arbitrary* $\boldsymbol{y}_0 \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$. *Suppose that there exists a time interval* $I := (t_1, t_2) \subseteq [0, T']$ *such that* $m(t) = 0$, *for all* $t \in I$. *Then there exists* $v \in [0, 1]$ *such that* $u^*(t) \equiv v$ *for all* $t \in I$. *Moreover,* v *is unique.*

Proof. Since $m(t) = 0$,

$$\boldsymbol{\lambda}(t) \propto JB\boldsymbol{x}^*(t), \qquad t \in I. \tag{20}$$

Also $\dot{m}(t) = 0$, $t \in I$, so (10) yields

$$\boldsymbol{\lambda}^T(t)(\operatorname{ad}_A B)\boldsymbol{x}^*(t) = 0, \qquad t \in I. \tag{21}$$

Combining (20) and (21) yields

$$(\boldsymbol{x}^*(t))^T M \boldsymbol{x}^*(t) = 0, \qquad t \in I. \tag{22}$$

where $M := B^T J^T (\operatorname{ad}_A B) + (B^T J^T (\operatorname{ad}_A B))^T$. We now show that on the singular interval $\boldsymbol{x}^*(t)$ remains on a line, i.e. there exists a function $c(\cdot) : [t_1, t_2] \to \mathbb{R} \setminus \{0\}$ such that

$$\boldsymbol{x}^*(t) = c(t)\boldsymbol{x}^*(t_1), \qquad t \in I. \tag{23}$$

We consider several cases.

Case 1. Suppose that the symmetric matrix M is sign-definite. Then the only solution of (22) is $\boldsymbol{x}^*(t) = \mathbf{0}$, which is a contradiction since $\boldsymbol{x}_0 \neq \mathbf{0}$.

Case 2. Suppose that M is singular. Suppose first that $M = 0$. Denote the elements of the matrix A by $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$. Since invertible linear transformations of the state space do not change the structure of optimal arcs, we may assume that B is either diagonal or in Jordan form. In the latter case: $B = \begin{pmatrix} b_1 & 1 \\ 0 & b_1 \end{pmatrix}$ and $\operatorname{ad}_A B = \begin{pmatrix} a_3 & a_4 - a_1 \\ 0 & -a_3 \end{pmatrix}$. A calculation shows that $M = 0$ implies that $b_1 a_3 = b_1(a_1 - a_4) - a_3 = 0$. If $a_3 = 0$ and $a_1 = a_4$, then $\operatorname{ad}_A B = 0$, and this contradicts the fact that $m(t) \equiv 0$ (see Example 1 above). Thus, $a_3 = b_1 = 0$, $a_4 - a_1 \neq 0$, so $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\operatorname{ad}_A B = \begin{pmatrix} 0 & a_4 - a_1 \\ 0 & 0 \end{pmatrix}$.

This yields $m(t) = \lambda_1(t)x_2^*(t)$, so $\lambda_1(t)x_2^*(t) = 0$. If $\lambda_1(t) = 0$, then it is possible to show that the last assertion in Theorem 3 does not hold. This contradiction implies that $x_2^*(t) = 0$, so (23) indeed holds.

The analysis in the case where B is diagonal is similar, and is therefore omitted.

Now consider the case where $M \neq 0$. Since M is singular and $M \neq 0$, the solution of (22) is indeed a line.

Case 3. Suppose that M is not singular and not sign-definite. It is straightforward to show the solution set of $\boldsymbol{x}^T M \boldsymbol{x} = 0$ is two lines that intersect at the origin.

Summarizing, on the singular interval $\boldsymbol{x}^*(t)$ satisfies (23). This yields

$$\dot{\boldsymbol{x}}^*(t) = (A + u^*(t)B)c(t)\boldsymbol{x}^*(t_1) = \dot{c}(t)\boldsymbol{x}^*(t_1), \tag{24}$$

so $\boldsymbol{x}^*(t_1)$ is an eigenvector of the matrix $A + u^*(t)B$ for all $t \in I$. In particular, there must exist $v \in [0, 1]$ and $p \in \mathbb{R}$ such that

$$(A + vB)\boldsymbol{x}^*(t_1) = p\boldsymbol{x}^*(t_1). \tag{25}$$

Combining this with (24) yields

$$c(t)(u^*(t) - v)B\boldsymbol{x}^*(t_1) = (\dot{c}(t) - pc(t))\boldsymbol{x}^*(t_1). \tag{26}$$

Suppose for a moment that $\boldsymbol{x}^*(t_1)$ is an eigenvector of B. It then follows from (25) that $\boldsymbol{x}^*(t_1)$ is also an eigenvector of A. This implies that $(\operatorname{ad}_A B)\boldsymbol{x}^*(t_1) = \mathbf{0}$ and, inductively, that $C\boldsymbol{x}^*(t_1) = \mathbf{0}$ for any matrix C that is a Lie-product of A and B. It is easy to see that this implies that $L_0(\boldsymbol{y}^*(t_1)) = \operatorname{span}\{B\boldsymbol{y}^*(t_1)\}$. But then $m(t_1) = 0$ implies that $\boldsymbol{\eta}(t_1)$ is trivial on $L_0(\boldsymbol{y}^*(t_1))$, and this contradicts Theorem 3.

We conclude that $\boldsymbol{x}^*(t_1)$ is not an eigenvector of B. Then the value v for which (25) holds is unique, and (26) implies that $u^*(t) = v$ for all $t \in I$. □

The next example demonstrates a case where no bang-bang control is optimal and, therefore, the optimal control must indeed be singular.

Example 2 Consider the linear switched system (4) with $A_0 = \begin{pmatrix} \gamma & 1 \\ 0 & \gamma \end{pmatrix}$, $A_1 = A_0^T$, where $\gamma < 0$. Note that the A_is are Hurwitz and Metzler. A calculation yields $G := \exp(A_1)\exp(A_0) =$

$e^{2\gamma}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$. It is easy to prove by induction that for $n \geq 1$: $G^n = e^{2n\gamma}\begin{pmatrix} F_{2n-2} & F_{2n-1} \\ F_{2n-1} & F_{2n} \end{pmatrix}$, where F_n is the nth Fibonacci number. This implies that for $\boldsymbol{x}(0) = (1,1)^T$, the switched system admits a solution satisfying $\boldsymbol{x}(2n) = G^n\boldsymbol{x}(0) = e^{2n\gamma}\left(F_{2n}, F_{2n+1}\right)^T$. Now, it is well-known that F_n grows nearly as $\phi^n/\sqrt{5}$, where $\phi := (1+\sqrt{5})/2 \approx 1.618$ is the golden ratio. Choosing $\gamma < 0$ such that $e^{\gamma} = 1/\sqrt{\phi}$, yields that both components of $\boldsymbol{x}(2n)$ grow at least as fast as $\Theta(\phi^n)$. Thus, the switched system is clearly not GUAS.

Since A_0 and A_1 are Hurwitz, there clearly exists $\bar{\beta} \in \mathcal{KL}$ such that $|\exp(A_0 t)\boldsymbol{x}_0| \leq \bar{\beta}(|\boldsymbol{x}_0|, t)$, and $|\exp(A_1 t)\boldsymbol{x}_0| \leq \bar{\beta}(|\boldsymbol{x}_0|, t)$ for all $\boldsymbol{x}_0 \in \mathbb{R}^2$ and all $t \geq 0$. The matrices are also Metzler, so if there always exists an optimal control u^* that is bang-bang, Theorem 2 would imply that $R(T; \mathcal{U}, \boldsymbol{x}_0) = R(T; \mathcal{BB}_3, \boldsymbol{x}_0)$. Proposition 1 then implies that the corresponding control system is GAS, and so the switched system is GUAS. But we already showed that this is not true.

It is also straightforward to compute that, given $\boldsymbol{x}_0$ and γ as above, and using $A = A_0$, $B = A_1$, and $u \equiv 1/2$ in (1) yields components of $\boldsymbol{x}(2n)$ that grow as $\Theta(e^n)$. □

C. Junctions

We now consider optimal controls that are concatenations of bang-bang and singular arcs.

Proposition 6 *If* $u^* : [0, T'] \to [0,1]$ *is a time-optimal control for* (5) *then either* $u^* \in \mathcal{BP}$ *or* $u^* \in \mathcal{PC}_2$.

Proof. We consider two cases.

Case 1: Suppose that u^* contains no bang arcs. The (absolute) continuity of $m(t)$ implies that in this case $m(t) = 0$ for all $t \in [0, T']$. Proposition 5 implies that $u^*(t) \equiv v$ for all $t \in [0, T']$, so $u^* \in \mathcal{PC}_0 \subset \mathcal{PC}_2$.

Case 2: Suppose that u^* contains a bang arc. Without loss of generality, we assume that there exist $0 \leq t_1 < t_2 \leq T'$ such that $m(t) > 0$ for $t \in I := (t_1, t_2)$.

Case 2.1: I is strictly contained in $(0, T')$. In this case, we may assume that $m(t_1) = m(t_2) = 0$, with $0 < t_1 < t_2 < T$. Let $s := t - t_1$. Eqs. (17) and (15) imply that

$$\begin{aligned} m(t) &= -g(-s)\boldsymbol{\lambda}^T(t_1)(\operatorname{ad}_A B)\exp(As)\boldsymbol{x}^*(t_1) \\ &= -g(-s)\boldsymbol{\lambda}^T(t_1)(\operatorname{ad}_A B)(f(s)I + g(s)A)\boldsymbol{x}^*(t_1). \end{aligned}$$

Suppose that $\dot{m}(t_1) = 0$, that is, $\boldsymbol{\lambda}^T(t_1)(\mathrm{ad}_A B)\boldsymbol{x}^*(t_1) = 0$. Then

$$m(t) = -g(-s)g(s)\boldsymbol{\lambda}^T(t_1)(\mathrm{ad}_A B)A\boldsymbol{x}^*(t_1). \tag{27}$$

At the switching point t_2 this yields: $0 = g(-t_{21})g(t_{21})\boldsymbol{\lambda}^T(t_1)(\mathrm{ad}_A B)A\boldsymbol{x}^*(t_1)$. Now consider that if $\boldsymbol{\lambda}^T(t_1)(\mathrm{ad}_A B)A\boldsymbol{x}^*(t_1) = 0$, then (27) implies that $m(t) \equiv 0$, for all $t \in I$, which is a contradiction. Therefore, $g(-t_{21})g(t_{21}) = 0$. Since $t_{21} > 0$, Eq. (16) implies that $p < 0$ and that $\sin(\sqrt{-p}t_{21}) = 0$. Then, arguing as in the proof of Proposition 3, we conclude that $u^*(t) = 0$ for all t, but this contradicts the fact that t_2 is a switching point.

Hence, $\dot{m}(t_1) \neq 0$. Similarly, it is possible to prove that $\dot{m}(t_2) \neq 0$. Since $m(t)$ is absolutely continuous this implies that t_1 (t_2) is the upper (lower) bound of another bang arc. Thus, u^* is composed of a concatenation of bang arcs, and either $u^* \in \mathcal{PC}_2$ or, by Remark 2, $u^* \in \mathcal{BP}$.

Case 2.2: Suppose that no bang arc is strictly contained in $[0, T']$. Thus, if (t_1, t_2) is a bang arc then either $t_1 = 0$ or $t_2 = T'$. The most general case possible is that we have two bang arcs: one on $(0, \tau_1)$ and the second on (τ_2, T'), with $0 < \tau_1 < \tau_2 < T'$, and the interval (τ_1, τ_2) does not contain any bang arc. It follows that $u^*(t) \equiv v$ for $t \in (\tau_1, \tau_2)$, so in this case $u^* \in \mathcal{PC}_2$. This completes the proof of Proposition 6. $\square$

V. PROOF OF THE MAIN RESULTS

We can now prove our main results. Fix arbitrary $\boldsymbol{x}_0 \in \mathbb{R}^2$, $T \geq 0$, and $u \in \mathcal{U}$. It follows from Propositions 2 and 6 that there exists $w \in (\mathcal{BB}_0 * \mathcal{BP}) \cup (\mathcal{BB}_0 * \mathcal{PC}_2)$ such that $\boldsymbol{x}(T; u, \boldsymbol{x}_0) = \boldsymbol{x}(T; w, \boldsymbol{x}_0)$. This completes the proof of Theorem 1. $\square$

Now suppose that there exists a proper cone $C \subset \mathbb{R}^2$ that is an invariant set for both $\dot{\boldsymbol{x}} = A\boldsymbol{x}$ and $\dot{\boldsymbol{x}} = (A + B)\boldsymbol{x}$. Recall that if u^* is a time-optimal bang-bang control for (5), with at least three switches $t_1 < t_2 < t_3$, then

$$\boldsymbol{x}^*(t_3) = k\boldsymbol{x}^*(t_1), \qquad k \neq 0. \tag{28}$$

By shifting the time axis, if necessary, we may assume that $t_1 > 0$.

For $i = 1, 2$, denote $l_i := \{c\boldsymbol{x}^*(t_i) : c \in \mathbb{R}\}$, that is, the line passing through $\mathbf{0}$ and $\boldsymbol{x}^*(t_i)$. Assume, without loss of generality, that the trajectories of $\dot{\boldsymbol{x}}^* = A\boldsymbol{x}^*$ cross the line l_1 in the clockwise direction (CWD). Let $S_{12} \subset \mathbb{R}^2$ denote the open cone enclosed between the lines l_1 and l_2 as shown in Fig. 1.

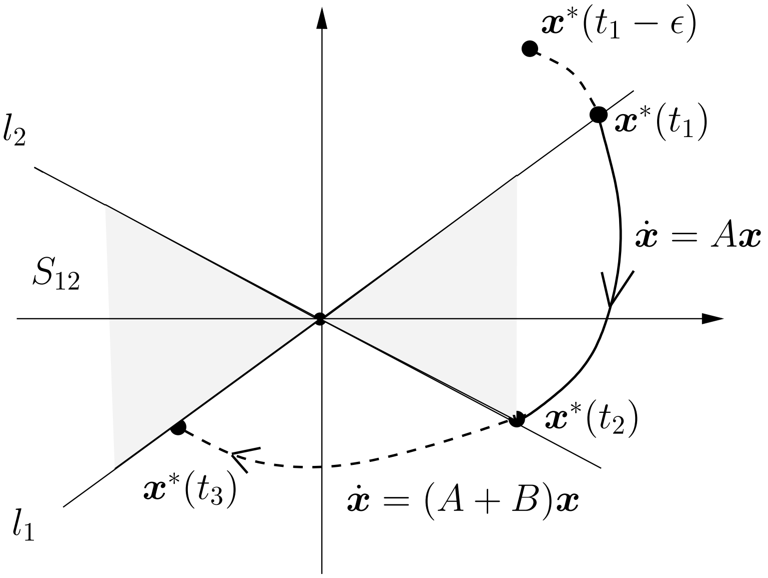

Fig. 1. An optimal bang-bang trajectory for the case where $\boldsymbol{x}^*(t_3) = k\boldsymbol{x}^*(t_1)$, with $k < 0$.

It follows from the homogeneity of the system that every point on l_i is a switching point. Furthermore, it follows that the optimal control is actually state-dependent and that if $\boldsymbol{x}^1, \boldsymbol{x}^2 \in \mathcal{S}_{12}$, then $u^*(\boldsymbol{x}^1) = u^*(\boldsymbol{x}^2)$. Assume, without loss of generality, that $u^*(\boldsymbol{x}) = 0$ for all $\boldsymbol{x} \in S_{12}$.

Since $t_1 > 0$ is a switching point, there exists $\epsilon > 0$ such that $\dot{\boldsymbol{x}}^* = (A+B)\boldsymbol{x}^*$ for $t \in (t_1 - \epsilon, t_1)$. This implies that the trajectories of $\dot{\boldsymbol{x}}^* = (A+B)\boldsymbol{x}^*$ also cross l_1 in the CWD. It follows from (28) that $\boldsymbol{x}^*(t_3) \in l_1$, so at time t_3 the trajectory $\boldsymbol{x}^*(t)$ also crosses l_1 in the CWD.

We now consider two cases. If $k > 0$ then the curve $\gamma := \{\boldsymbol{x}^*(t) : t \in (t_1, t_3]\}$ completes a whole turn around the origin in the CWD. Thus, there exist times $\tau_1, \tau_2 \in [t_1, t_3]$ such that $\boldsymbol{x}^*(\tau_1) \in C$ and $\boldsymbol{x}^*(\tau_2) \in -C$. But, since both C and $-C$ are invariant sets for the dynamics, and $C \cap (-C) = \{\boldsymbol{0}\}$, this is a contradiction.

The case $k < 0$ (see Fig. 1) cannot hold because $\boldsymbol{x}^*(t_1) \in C$ and invariance implies $\boldsymbol{x}^*(t_3) \in C$, but (28) with $k < 0$ says $\boldsymbol{x}^*(t_3) \in -C$. We conclude that $\boldsymbol{x}^*(t_3) \in C \cap (-C) = \{\boldsymbol{0}\}$, a contradiction.

Summarizing, any bang-bang time-optimal control cannot include more than two switching points. Thus, any time-optimal control satisfies $u^* \in (\mathcal{BB}_2 \cup \mathcal{PC}_2)$. Using Proposition 2 completes the proof of Theorem 2. $\square$

ACKNOWLEDGMENTS

We are grateful to the anonymous reviewers for their detailed and constructive comments.

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