Perspectives and Results on the Stability and Stabilizability of Hybrid Systems

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Invited Paper

This paper introduces the concept of a hybrid system and some of the challenges associated with the stability of such systems, including the issues of guaranteeing stability of switched stable systems and finding conditions for the existence of switched controllers for stabilizing switched unstable systems. In this endeavor, this paper surveys the major results in the (Lyapunov) stability of finite-dimensional hybrid systems and then discusses the stronger, more specialized results of switched linear (stable and unstable) systems. A section detailing how some of the results can be formulated as linear matrix inequalities is given. Stability analyses on the regulation of the angle of attack of an aircraft and on the PI control of a vehicle with an automatic transmission are given. Other examples are included to illustrate various results in this paper.

Keywords**—***Hybrid systems, linear matrix inequalities, stability, stabilizability, switched systems.*

I. INTRODUCTION

Loosely speaking, hybrid systems consist of continuous time (CT) and/or discrete time (DT) processes interfaced with some logical or decision-making process. The continuous/discrete time (C/DT) component might consist of differential/difference equations or continuous/discrete time state models. The logical/decision-making (LDM) component might be a finite automaton or a more general discrete event system [1]. The C/DT processes affect the state transitions of the LDM, and the LDM processes affect the dynamic motions of the C/DT processes [2], [3]. Examples include disk drives [56], stepper motors [57], the on–off behavior of a furnace [24], [25], a VSTOL aircraft [58], etc.

Example 1: For a simplified model of a manual transmission [57], let x_1 denote the ground position of a car, x_2 the engine RPM, and control $u \in [0,\, 1]$ the throttle position. The simplified (hybrid) dynamics are

$$\dot{x}_1 = x_2 \tag{1.1a}$$

$$\dot{x}_2 = \frac{1}{1+\beta(p)}\left[-\alpha(x_2) + u\right] \tag{1.1b}$$

where $\alpha(x_2) > 0$ when $x_2 > 0, p \in \{1,\, 2,\, 3,\, 4\}$ denotes the gear shift position, and $\beta(p)$ is some function of p. Equation (1.1b) comes in four flavors depending on the decision of the external agent, who controls "p." Hence, the hybrid system consists of four distinct dynamic motions (four distinct CT state models) indexed by p and concatenated in some way to produce an overall dynamic motion. □

This paper overviews current results on the stability of hybrid systems and, within the narrow confines of switching among a finite set of possible motions, the stabilization of hybrid systems. For other overviews, see [4]–[8] and [26]. The following two examples motivate the rich challenges and unforeseen surprises intrinsic to the stability of hybrid systems.

Example 2 [49]: Consider the autonomous state dynamics $\dot{x}(t) = A_{p(t)}x(t)$ where $x = [x_1,\, x_2]^T \in R^2, p \in \{1,\, 2\}$ and

$$A_1 = \begin{bmatrix} -1 & -100 \\ 10 & -1 \end{bmatrix}; \qquad A_2 = \begin{bmatrix} -1 & 10 \\ -100 & -1 \end{bmatrix}.$$

Both A_1 and A_2 are stable, having identical eigenvalues $\lambda_{1,\,2} = -1 \pm j\sqrt{1000}$. Define the switching function $p(t)$ as follows:

$$p(t^+) = \begin{cases} 1, & \text{if } p(t) = 2 \text{ and } x_2(t) = -\dfrac{1}{k}\, x_1(t) \\ 2, & \text{if } p(t) = 1 \text{ and } x_2(t) = kx_1(t). \end{cases} \tag{1.2}$$

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For any given initialization, this function specifies a rule with memory for switching the dynamic motion of the system between A_1 and A_2. For $k = -0.2$ and any $x(0) \neq 0$, the state trajectory diverges to ∞ (Fig. 1), showing that switching between two asymptotically stable systems can produce an unstable trajectory. □

Switching between two asymptotically stable systems as above can occur in the control of the longitudinal dynamics of an aircraft with constrained angle of attack (see Example 4). Two questions arise: a) what classes of stable systems admit a stable state trajectory for all switching sequences and b) what switching sequences always result in stable trajectories? If there exists a common Lyapunov function $V(x)$ for a set of stable A-matrices, the resulting system is stable for all switching sequences [26], which answers question a). A partial answer to question b) is intuitive: If switching among asymptotically stable systems is slow enough, one would expect a stable response. Stability here is characterized by a traditional Lyapunov function $V(x)$ that measures the system energy. Mathematically, $V(\cdot)$ is continuous and differentiable,[1] $V(0) = 0$, and $V(x) > 0$ if $x \neq 0$. Further, if $\dot{V} < 0$, $x \neq 0$, then the state $x(t)$ will converge to zero, implying local stability (global stability if $V(\cdot)$ is radially unbounded) [91]–[93].

The fact that a common Lyapunov function guarantees stability for arbitrary switching has led researchers to search for conditions under which a common Lyapunov function exists. See, e.g., [27]. For switching between two stable linear systems in the plane, necessary and sufficient conditions for the existence of a common quadratic Lyapunov function have been derived [28]. For families of linear systems (indexed by a compact set), various algebraic properties of the family have proven to be sufficient. For example, if all of the matrices commute or have a solvable Lie algebra, then a common quadratic Lyapunov function exists and the system is stable under arbitrary switching. See [26] for more details and [29] for a generalization. Finally, sufficient conditions for a common Lyapunov function of a "quasi-quadratic" form has been generalized to nonlinear systems [9].

The on–off behavior of a furnace exemplifies switching between a stable system (off) and an unstable system (on) [24]. Under certain conditions, if a furnace locks to the on-mode, the temperature outside the furnace will continuously rise, an unstable behavior. An even more complex problem, and one dual to the problem shown in Example 2, is that of switching between two unstable systems to produce a stable trajectory, as illustrated in Example 3.

[1]Continuous and differentiable is the classical and customary assumption on the Lyapunov function $V(\cdot)$ due apparently to the fact that system dynamics have often been described by continuous and differentiable vector fields. This, however, is not required. See, e.g., [94] and [95]. Work in hybrid systems generally avoids this assumption as in Theorem 4.1 in [2] and, of course, throughout this paper. More recently, because one may be switching among unstable systems, the phrase "multiple Lyapunov functions" has appeared.

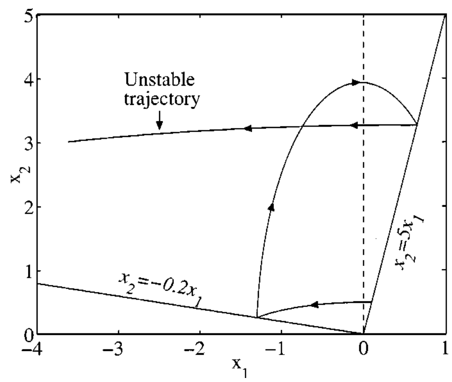

Fig. 1. Unstable state trajectory of switched stable systems.

Example 3 [30], [50]: Again, consider the autonomous state dynamics $\dot{x}(t) = A_{p(t)}x(t)$, where $p(t) \in \{1, 2\}$ and

$$A_1 = \begin{bmatrix} 0 & 10 \\ 0 & 0 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1.5 & 2 \\ -2 & -0.5 \end{bmatrix} \tag{1.3a}$$

and

$$p(t) = \begin{cases} 1, & \text{if } p(t^-) = 2 \text{ and } x_2(t) = -0.25x_1(t) \\ 2, & \text{if } p(t^-) = 1 \text{ and } x_2(t) = 0.5x_1(t). \end{cases} \tag{1.3b}$$

Both systems are unstable as the eigenvalues of A_1 lie at zero and those of A_2 lie at $0.5 \pm j\sqrt{3}$. The phase-plane portraits of the individual systems are shown in Fig. 2.

These phase-plane trajectories can be pieced together to produce a stable state trajectory as shown in Fig. 3.

A stability analysis of this system serves to motivate many ideas contained in this paper. For the system structures A_1 and A_2, define quadratic Lyapunov-like functions $V_1(x) = x^T P_1 x$ and $V_2(x) = x^T P_2 x$, where

$$P_1 = \begin{bmatrix} 0.46875 & -1.875 \\ -1.875 & 15 \end{bmatrix}, \qquad P_2 = \begin{bmatrix} 1 & 1.2 \\ 1.2 & 1.6 \end{bmatrix}.$$

Quadratic Lyapunov-like functions having negative definite derivative in some region of the state space always exist provided $A_i \neq \beta I$ with $\beta \geq 0$ [30]. As such, define the regions $\Omega_i = \{x | \dot{V}_i(x) = x^T(A_i^T P_i + P_i A_i)x \leq 0\}, i = 1, 2$. Thus, in Ω_i, the system energy, defined by $V_i(x)$, decreases. One may also verify that $\Omega_1 \cup \Omega_2$ covers the entire state space R^2. Since the energy is decreasing in Ω_i, the state moves closer to the origin within Ω_i, as measured by the ellipsoidal level sets defined by the functions $V_i(x)$. Placing the $V_i(x(t))$ [$x(t)$ being the actual system state trajectory] side by side results in an overall piecewise continuous and piecewise quadratic Lyapunov-like function whose energy ultimately decreases to zero as per Fig. 4. □

The ideas a) that the union of the Ω-regions covers the state space; b) that in each Ω-region there are one (or more) (quadratic) Lyapunov-like functions whose energy decreases along the system state trajectories; and c) that these possibly multiple Lyapunov-like functions can be pieced together in some way to produce a global (nontraditional) Lyapunov function whose overall energy decreases to zero along the system state trajectories underlie the main results of this paper.

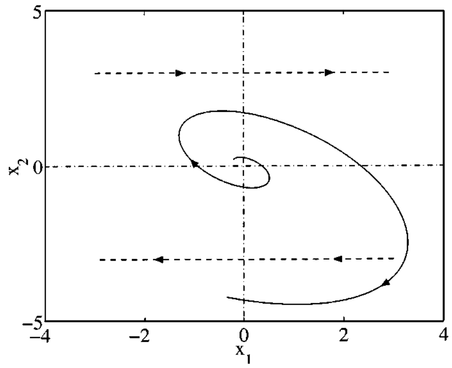

Fig. 2. Dashed line is phase-plane portrait if $\dot{x} = A_1 x$; solid line is phase-plane portrait of $\dot{x} = A_2 x$.

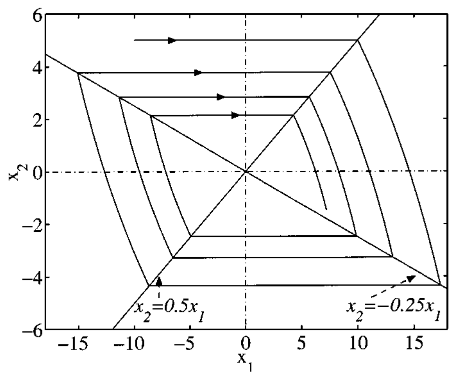

Fig. 3. State trajectory resulting from switching between A_1 and A_2; the lines $x_2 = 0.5x_1$ and $x_2 = -0.25x_1$ lie in $\Omega_1 \cap \Omega_2$.

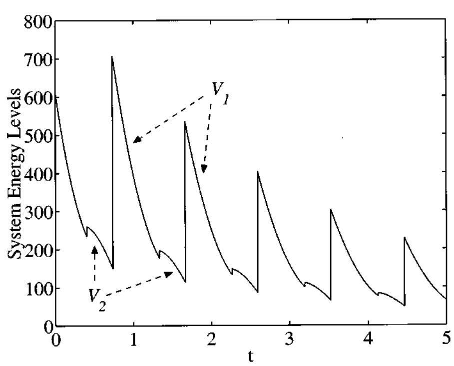

Fig. 4. Energy profile of the switched system with trajectories shown in Fig. 3.

Example 3 illustrates the stability and stabilizability of two linear time invariant systems with state spaces in the plane R^2. An exhaustive analysis of various properties of switched systems with state vectors in R^2 was set forth in Loparo *et al.* in [31] and [32]. More recently, using a geometric approach, Xu and Antsaklis [33], [34] were able to obtain necessary and sufficient conditions for asymptotic stabilizability of two-switched LTI systems with state spaces in R^2. Characterizing the stability and stabilizability of switching among families of nonlinear systems presents a much more challenging task. For the most general results in this area, see Branicky [1], [6], [10], Michel *et al.* [2], [3], [8], and Pettersson and Lennartson [11], [12], [49]. The substance of these results is described in Section III.

This paper is organized as follows. Section II briefly introduces the model of a hybrid system used throughout the paper, while illustrating the representation by the longitudinal control of an aircraft and the PI cruise control of a vehicle with an automatic transmission. Careful stability analyses of these examples are set forth in Section VI. In between, in Section III, we overview the development of the main stability results for finite-dimensional hybrid systems. These results are specialized for switched linear systems in Section IV. In Section V, a numerical approach for the stabilization of such systems is formulated and set forth as the solution of a linear matrix inequality (LMI). Example 3 is revisited and solved via the LMI approach, also in Section V.

II. A Hybrid System Model

Two of the earliest hybrid system models were those of Witsenhausen [63] and Tavernini [64]. However, in the last decade or so, hybrid system modeling has evolved dramatically beginning with Peleties and DeCarlo [50], [65]–[68] and continuing with Stiver *et al.* [69], [70], Alur *et al.* [71], Nerode *et al.* [72], [73], Dogruel–Ozguner [74], Branicky [51], [75], [76], and, more recently, Pettersson *et al.* [49], [77], [78]. A review of these models is beyond the scope of this paper. Nevertheless, the pertinent fruit of the modeling evolution can be summarized by the dynamics

$$\dot{x}(t) = f(x(t), p(t), u(t)) \tag{2.1a}$$

$$y(t) = g(x(t), p(t), u(t)) \tag{2.1b}$$

$$p(t^+) = \varphi(x(t), p(t), u(t), \sigma(t)) \tag{2.1c}$$

where $x(t) \in R^n$ is the continuous-time state, $u(t)$ can be either a continuous control input or some external (reference or disturbance) signal to the continuous-time part of the system, $p(t) \in \{1, \cdots, M\}$ represents a discrete state that indexes the vector fields $f(\cdot, \cdot, \cdot)$ that determine $\dot{x}$, $\sigma(t)$ is a "discrete event" (possibly control) input, and $\varphi(\cdot, \cdot, \cdot, \cdot)$ is a discontinuous transition function behind which lies a whole set of undescribed logical and/or discrete-event system dynamics. In many cases, $p(t)$ serves as the "control," as in Example 1. A special case of model (2.1) is the *autonomous model*, in which $u(t)$ and $\sigma(t)$ are not present, as in Examples 1–3. In particular, for Examples 2 and 3, $f(x(t), p(t), u(t)) = A_{p(t)}$, $p(t) \in \{1, 2\}$, with the transition functions defined in (1.2) and (1.3b). The autonomous model also results when $u(t)$ and $\sigma(t)$ are explicit (feedback) functions of the state.

Example 4: The highly simplified longitudinal dynamics of an aircraft can take the form [51], [59]

$$\begin{bmatrix} \dot{q} \\ \dot{\alpha} \end{bmatrix} = \dot{x} = f(x, u, p) = \begin{bmatrix} -1 & -10 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} q \\ \alpha \end{bmatrix} + \begin{bmatrix} -1 \\ 0.1 \end{bmatrix} u_{p(t)} \tag{2.2a}$$

where $\alpha \leq \alpha_{\lim}$ is the constrained angle of attack and q is the pitch rate. The output equation is

$$\begin{bmatrix} \alpha \\ n_z \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -300 \end{bmatrix} \begin{bmatrix} q \\ \alpha \end{bmatrix} + \begin{bmatrix} 0 \\ 30 \end{bmatrix} u_{p(t)} \tag{2.2b}$$

where $p(t) \in \{1, 2\}$, n_z is the normal acceleration, and the control variable $u_{p(t)}$ is the angle of the elevator measured down from the horizontal with the aircraft. The control objective is twofold: track the pilot's reference normal acceleration while maintaining the safety constraint that the angle of attack must be less than $\alpha_{\rm lim}$. To simultaneously achieve both objectives (to the extent possible), we define a switched "max control law"

$$p(t^+) = \varphi(x(t), u(t), p(t)) = \arg\max_i(u_i) \qquad (2.3)$$

where

$$\begin{aligned} u_1 &= -Fx + k_1\alpha_{\rm lim} \\ u_2 &= -Gx + k_2 r(t). \end{aligned} \qquad (2.4)$$

Here, u_1 is the output of a controller designed to stabilize the aircraft about $\alpha_{\rm lim}$, and u_2 is a control designed to make n_z track $r(t)$. Roughly, the max control law acts to track the pilot's reference using the elevator except when to do so would cause the safety constraint to be violated. ∇

Example 5 [49]: The simplified dynamics of a car (mass m) with an automatic transmission having velocity v on a road inclined at an angle α is

$$\dot{v} = -\frac{k}{m}v^2\,\mathrm{sign}(v) - g\sin(\alpha) + \frac{G_{p(t)}}{m}T \qquad (2.5a)$$

$$\omega = G_{p(t)}v \qquad (2.5b)$$

where the discrete state $G_{p(t)} \in \{G_1, G_2, G_3, G_4\}$, $G_1 > G_2 > G_3 > G_4$ are the transmission gear ratios normalized by the wheel radius R, k is an appropriate constant, ω is the angular velocity of the motor, and T is the torque generated by the engine, an input to the model. The discrete state transition function is

$$p(t^+) = \begin{cases} i+1, & \text{if } p(t) = i \neq 4 \text{ and } v = \dfrac{1}{G_i}\,\omega_{\rm high} \\ i, & \text{if } p(t) = i+1 \geq 2 \text{ and } v = \dfrac{1}{G_{i+1}}\,\omega_{\rm low} \end{cases} \qquad (2.6)$$

where $\omega_{\rm high}$ and $\omega_{\rm low}$ are preset angular velocities of the engine, as illustrated by the automaton of Fig. 5.

A PI cruise controller (of the torque) that must also compensate for the nonlinear load forces is

$$T = T_P + T_I + \frac{k}{G_{p(t)}}v^2\,\mathrm{sign}(v) \qquad (2.7)$$

for a reference velocity $v_{\rm ref}$ and a proportional control $T_p = K_{p(t)}(v_{\rm ref} - v)$. This leads to combined/reduced vehicle-cruise controller dynamics

$$\dot{v} = \frac{G_{p(t)}}{m}\left(K_{p(t)}(v_{\rm ref} - v) + T_I\right) - g\sin(\alpha) \qquad (2.8a)$$

$$\dot{T}_I = \frac{K_{p(t)}}{T_R}(v_{\rm ref} - v). \qquad (2.8b)$$

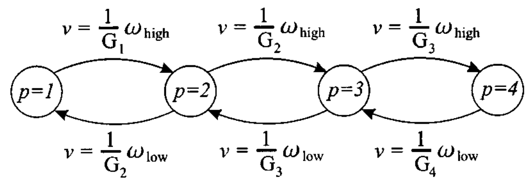

Fig. 5. Automaton illustrating discrete state transistors.

The constant T_R is chosen to balance fast convergence with small overshoot; the discrete gains $K_{p(t)} \in \{K_1, K_2, K_3, K_4\}$ are chosen to insure a smooth ride and satisfy

$$G_iK_i = G_{i+1}K_{i+1}. \qquad (2.9a)$$

The initial condition on (2.8b) is a) reset to zero for new $v_{\rm ref}$ inputs and b) for any change in the discrete state $p(t)$ at, say, t_k, the state $T_I(t_k^+)$ is reset discontinuously (a so-called state jump) so that

$$G_{p(t_k^-)}T_I(t_k^-) = G_{p(t_k^+)}T_I(t_k^+) \qquad (2.9b)$$

also to ensure a smooth ride. $\square$

A stability analysis of Examples 4 and 5 is given in Section VI.

III. General Stability Results for Hybrid Systems

Classical Lyapunov stability theory has been the Clydesdale of system stability for the past century. Demonstrating stability depends on the existence and/or construction of an appropriate (continuous and differentiable) Lyapunov (energy) function that may not exist (as per Example 3) or, when it does exist, may be difficult to construct. For hybrid systems, demonstrating existence and/or constructing a classical Lyapunov function only worsens, at best. Yet, the intrinsic discontinuous nature of a hybrid system strongly suggests using multiple Lyapunov-like functions concatenated together to produce a nontraditional (piecewise continuous and piecewise differentiable) Lyapunov function as in [1]–[5], [13], [24], [26], [30], [35], [36], [49]–[52], and [57]. Using multiple Lyapunov functions (MLF's) to form a single nontraditional Lyapunov function offers much greater freedom and infinitely more possibilities for demonstrating stability, for constructing a nontraditional Lyapunov function, and for achieving the stabilization of the hybrid system (2.1), which we now restrict to the special (autonomous) form

$$\dot{x}(t) = f(x(t), p(t)) \equiv f_{p(t)}(x(t)) \qquad (3.1)$$

where $p(t) \in \{1, \cdots, M\}$. Further, it is assumed that $p(t)$ is piecewise continuous (from the right), implying that there are only a finite number of switches per unit time.[2]

[2]B definition, piecewise continuous means a finite number of discontinuities per unit interval with well-defined left- and right-hand limits at each discontinuity point. "From the right" further requires continuity from the right. Theoretically, one can allow for infinitely fast switching, which brings the problem to the realm of hybrid sliding mode control. Practically speaking, this is to be avoided because of the finite bandwidth of actuators and the undesirability of exciting (unmodeled) high-frequency dynamics of the system.

One of the early results of hybrid system stability for linear switched systems was developed by Peleties [5], [50] for system (3.1) when $f_i(x) = A_i x$. Briefly, define a family of Lyapunov-like functions $\{V_i,\, i = 1, \cdots, M\}$, each associated with the vector field $f(x,\, i) = f_i(x)$. A *Lyapunov-like function* for the system $\dot{x} = f_i(x)$ and equilibrium point $\overline{x} \in \Omega_i \subset R^n$ (the state space) is a real-valued function $V_i(x)$ (with continuous partial derivatives) defined over the region Ω_i satisfying the conditions:

i) *positive definiteness:* $V_i(\overline{x}) = 0$ and $V_i(x) > 0$ for $\overline{x} \neq x \in \Omega_i$; often $\overline{x} = 0$;

ii) *negative definite derivative:* for $x \in \Omega_i$

$$\dot{V}_i(x) = \frac{\partial V_i(x)}{\partial x}\, f_i(x) \leq 0. \tag{3.2}$$

Note that Ω_i is precisely the set of x for which (3.2) holds.

Theorem 3.1: Suppose that $\bigcup_k \Omega_k = R^n$, the state space. For $i < j$, let $t_i < t_j$ be switching times for which $p(t_i) = p(t_j)$, and suppose there exists $\gamma > 0$ such that

$$V_{p(t_j)}\,(x(t_{j+1})) - V_{p(t_i)}\,(x(t_{i+1})) \leq -\gamma \, \|x(t_{i+1})\|^2. \tag{3.3}$$

It follows that system (3.1), with $f_{p(t)}(x) = A_{p(t)} x$ and switching function $p(t)$, is globally asymptotically stable [5], [26].

Condition (3.3) of Theorem 3.1 is illustrated in Fig. 4. Branicky set forth the first nonlinear generalization [10], [51]; other significant extensions are in [8], [14], and [15].

Theorem 3.2 [10], [51]: Given the M-switched nonlinear system (3.1), suppose each vector field f_i has an associated Lyapunov-like function V_i in the region Ω_i, each with equilibrium point $\overline{x} = 0$, and suppose $\bigcup_i \Omega_i = R^n$. Let $p(t)$ be a given switching sequence such that $p(t)$ can take on the value i only if $x(t) \in \Omega_i$, and in addition

$$V_i(x(t_{i,\,k})) \leq V_i(x(t_{i,\,k-1})) \tag{3.4}$$

where $t_{i,\,k}$ denotes the kth time that vector field f_i is "switched in," i.e., $p(t^-_{i,\,k}) \neq p(t^+_{i,\,k}) = i$. Then (3.1) is Lyapunov stable.

Fig. 6 illustrates the meaning of condition (3.4) and a more general result due to Ye *et al.* [2], [8], [15]. Beginning with different assumptions, this more general result assumes a so-called *weak Lyapunov function* for V_i, in which condition (3.2) is replaced by

$$V_i(x(t)) \leq h(V_i(x(t_j))), \qquad t \in (t_j,\, t_{j+1}) \tag{3.5}$$

where h: $R^+ \to R^+$ is a continuous function that satisfies $h(0) = 0$, and where t_j is any switching instant when system i is activated, i.e., $i = p(t^+_j) \neq p(t^-_j)$. Here, the energy over each time interval for which system i is active remains bounded in the sense of condition (3.5), potentially allowing the system energy to increase in these ranges. Since $\Omega_i = \{x \,|(\partial V(x)/\partial x) f_i(x) \leq 0\}$, the result by Ye *et al.* permits f_i to be activated for $x \notin \Omega_i$.

Intuitively speaking, one might expect that the system energy could increase for short periods of time provided other vector fields, f_j, when activated, overcome this "small" increase. Thus, the relaxation of condition (3.2) allows greater freedom for the designer to prove stability and to construct stabilizing switching sequences. Michel *et al.* have developed and formulated their ideas and results in a quite general setting capable of handling the diverse motions/behaviors found in finite as well as infinite dimensional hybrid dynamical systems [2], [3], [8], [16]. Related issues with regard to systems with discontinuous jumps in the state (impulse effects) can be found in [17]–[19].

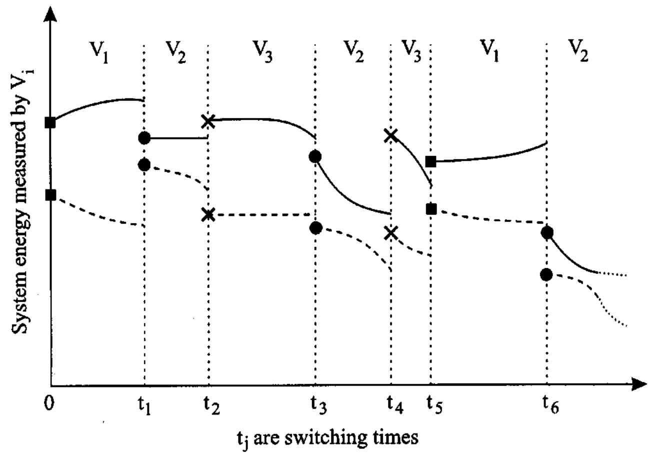

Fig. 6. Dashed line (lower curves) illustrates Theorem 3.2; solid line (upper curves) shows example of weakening condition (3.2)–(3.4); squares initiate V_1, circles initiate V_2, $\times$'s initiate V_3.

A further extension of the above results is formalized in Pettersson and Lennartson [11], [12], [49], where the one (V_i, Ω_i) pair for each vector field f_i is relaxed to allow multiple pairs $(V_{\rm ij}, \Omega_{\rm ij})$, $j = 1, \cdots, M_i$ for each vector field f_i. Note there is no requirement that $V_{\rm ij} \neq V_{\rm km}$. In addition, they have relaxed the assumptions of globally Lipschitz to locally Lipschitz and the condition that $f_i(\overline{x}) = 0$ for all i to $f_i(\overline{x}) = 0$ for only a subset of the vector fields. The proofs used in [1], [6], [8], [51], and [52] extend with some routine modifications by viewing each new pair $(V_{\rm ij},\, \Omega_{\rm ij})$ as a distinct discrete state. We will see applications of this idea in Section VI.

All of the above theorems can be strengthened to produce asymptotic and exponential stability [1], [2], [8], [11], [15], [49], [51], [52]. An example is Theorem 3.3.

Theorem 3.3 [49, Theorem 4.1]: If for each $x \in R^n$, $x \neq 0$, a vector field $f_i(\cdot)$ can be selected such that $x^T f_i(x) < 0$, and any resulting sliding motion dynamics are given by Filippov's convex combination definition [94], then the origin 0 of the closed-loop system is stable in the sense of Lyapunov. Additionally, if there exists a constant $\gamma > 0$ such that $2x^T f_i(x) \leq -\gamma x^T x$, then the origin 0 of the closed-loop system is exponentially stable.

Implicitly, the above theorem provides a methodology for switching between vector fields to achieve a stable trajectory. Other generalizations are possible. MLF theory can be extended to the case of multiple equilibria (a different equilibrium point for each vector field f_i) and the case where the index set is an arbitrary compact set [1].

Note that the restriction that there are a finite number of switches in finite time does not necessarily exclude sliding-like motions.[3] Indeed, sliding motions may be incorporated by defining each such motion and its associated equivalent dynamics [94] as an additional system to which we can switch. We then merely check the conditions of the theorems as before. For more on sliding in the context of hybrid systems, see [49], [53], [54], [96].

The general results presented above give sufficient conditions for hybrid stability. Such sufficient conditions often lead to powerful design rules, as per the sufficiency of Lyapunov's classical theorem [91]–[93] in adaptive control theory [97]. In the adaptive control area, plants sometimes have large uncertainty necessitating controllers designed for different admissible models; the hybrid control problem is to switch among these controllers to achieve stabilization [4], [26]. Various rules have been proposed for the stabilization of diverse hybrid systems, notably in [37], [49]–[55]. As an example of these results, Pettersson and Lennartson [11], [12], [49] proposed the following implementation of Theorem 3.3.

Min-Projection Strategy: For specific $x \in \bigcup_i \Omega_i$, choose the vector field $f_i(x)$ according to the criterion $i = \arg\min_k x^T f_k(x)$.

Thus, one strategy that may stabilize a hybrid system is to pick the vector field that causes maximal descent of a particular energy function(s), here $V = x^T x$. Another strategy to stabilize a hybrid system is to select vector fields according to the Lyapunov function with the smallest value as in [38] and [54]. A similar approach is given in [37]. Other viewpoints on the stabilization of hybrid systems can be found in [4], [13], [26], and [31]–[34].

The necessity of the existence of Lyapunov functions for switched and hybrid systems is being energetically explored. Such converse theorems can be found in [3] and [9].

IV. Stability and Stabilization of Linear Switched/Hybrid Systems

Switching among different system structures is an essential feature of many engineering control applications. Examples include gain scheduling, switched dc-to-dc power converters, pulse width modulation control, switched capacitor networks, and vibration suppression in structures using variable stiffness. See, for example, [60]–[62]. This section surveys some of the important results for linear M-switched systems, i.e., those satisfying

$$\dot{x}(t) = A_{p(t)} x(t) \tag{4.1a}$$
$$p(t^+) = \varphi(x(t), p(t)) \tag{4.1b}$$

for $p(t) \in \{1, 2, \cdots, M\}$. For this restricted class of hybrid systems, stronger results than those presented in Section III are possible. Particular choices for the switching function (4.1b) are set forth.

[3] For a sliding mode to exist on a lower dimensional manifold, the vector fields must point toward the manifold and drive any state deviations back to the manifold. Hence, motion in an invariant space is a sliding-like motion, but not necessarily a sliding mode.

A. Switching Among Stable A-Matrices

Suppose the A_i, $i = 1, \cdots, M$, are stable, i.e., all eigenvalues in open left-half complex plane. If all the A_i share a common Lyapunov function, $V(x) = x^T P x$, such that $\dot{V}(x) \le -x^T Q x$, $Q > 0$, then the system is exponentially stable for all switching sequences [26], [80]. *Exponential stability* means that there exist constants $\alpha_1 > 0$ and $\alpha_2 > 0$ such that

$$\|x(t)\| \le \alpha_1 e^{-\alpha_2 t} \|x(0)\|. \tag{4.2}$$

The above conditions on $V(x)$ are equivalent to $A_i^T P + P A_i \le -Q$ for all i. See [26] and [98] for more details. Finding such P and Q is an area of continued research. Approaches when the A_i commute can be found in [26], [28], and [39]. Lie-algebraic conditions (a major research direction) can be found in [29] and [40], [41]. However, the existence of a common quadratic Lyapunov function, although sufficient, is not necessary for stability [9]. Further, in [42], Hespanha and Morse show that if the on-average switching is "slow," then stability is guaranteed.

B. Switching Among Possibly Unstable A-Matrices

If at least one $A_i \in \mathcal{A} = \{A_1, \cdots, A_M\}$ has eigenvalues in the open right-half complex plane, then switching sequences $p(t)$ exist that destabilize system (4.1). Existence and construction of stabilizing switching sequences become critically important. If the set $\mathcal{A}$ contains an asymptotically stable A_i, then any switching sequence that eventually latches fixedly onto A_i stabilizes system (4.1). Hence, for the remainder of this section, we have the following.

Assumption 4.1: Each $A_i \in \mathcal{A}$ has eigenvalues in the closed right-half complex plane.

Thus, we seek an answer to the question of existence of switching sequences resulting in quadratic stability and, when possible, a feedback mechanism, $p(t^+) = \varphi(p(t), x(t))$, for stabilization. By *quadratic stability,* we mean that there exists a quadratic function $V(x) = x^T P x$ ($P > 0$, i.e., P is positive definite and symmetric), an $\varepsilon > 0$, and a switching sequence $p(t)$ such that

$$\dot{V}(x) = \left[\frac{\partial}{\partial(x)} V(x)\right] A_{p(t)} x < -\varepsilon x^T x. \tag{4.3}$$

When $M = 2$, i.e., $\mathcal{A} = \{A_1, A_2\}$, then the results of [30], [35], and [36] lead to the following.

Theorem 4.1: There exists a switching sequence $p(t)$ such that system (4.1) is quadratically stable iff there exists $\alpha \in (0, 1)$ such that

$$A_{\text{eq}} = \alpha A_1 + (1 - \alpha) A_2 \tag{4.4}$$

is a stability matrix, i.e., eigenvalues of A_{eq} lie in the open left-half complex plane.

Conditions for finding A_{eq} are in [35], with the caveat that general convex combinations are NP-hard [99], [100]. (A brute force approach is simply to plot the eigenvalues of $\alpha A_1 + (1-\alpha) A_2$ for $0 < \alpha < 1$.) When A_{eq} exists and α is known, then for arbitrary $Q > 0$, one can construct $P_{\rm eq} > 0$ [79], [86] satisfying

$$A_{\rm eq}^T P_{\rm eq} + P_{\rm eq} A_{\rm eq} = -Q$$

i.e., for appropriate switching $p(t)$, $V(x) = x^T P_{\rm eq} x$ satisfies condition (4.3) above.

In order to construct a piecewise continuous (from the right) $p(t)$, quadratically stabilizing system (4.1), define for $i = 1, 2$, positive definite matrices

$$Q_i = -(A_i^T P_{\rm eq} + P_{\rm eq} A_i), \tag{4.5}$$

and Ω-regions

$$\Omega_i = \left\{x | -x^T Q_i x < 0\right\}. \tag{4.6}$$

Here, we need to point out that $\Omega_1 \cup \Omega_2$ covers the state space and $\Omega_1 \cap \Omega_2 \neq 0$. Further, for sufficiently small (user chosen) $\varepsilon > 0$, define two (quadratic) switching surfaces

$$s_1(x) = x^T (Q_1 - \varepsilon Q_2) = 0 \tag{4.7a}$$
$$s_2(x) = x^T (Q_2 - \varepsilon Q_1) = 0. \tag{4.7b}$$

Theorem 4.2 [30]: Quadratic stabilization of system (4.1) results when using the following hybrid switching rule:

1) (initialization) at t_0 activate system $A_{p(t_0)}$ so that $x(t_0) \in \Omega_{p(t_0)}$;
2) $p(t^+) = 2$ when $p(t) = 1$ and $s_1(x) = 0$; (4.8)
3) $p(t^+) = 1$ when $p(t) = 2$ and $s_2(x) = 0$.

The switching rule of (4.8) stabilizes without generating a sliding mode since the hyperplanes defined by the $s_i(x) = 0$ do not overlap. For the strategies that may result in a sliding mode, see [35] and [36].

The necessity of condition (4.4) in Theorem 4.1 does not generalize for $M > 2$. Nevertheless, we can state the following theorem [30], [35].

Theorem 4.3: Let $\mathcal{A} = \{A_1, \cdots, A_M\}$ satisfy Assumption 4.1. A sufficient condition for the existence of $p(t)$ to produce a stable $x(t)$ satisfying system (4.1) is that there exist a stable convex combination of the A_i-matrices, i.e., there exist $\alpha_i > 0$, $\sum_i \alpha_i = 1$, such that

$$A_{\rm eq} = \sum_{i=1}^{M} \alpha_i A_i \tag{4.9}$$

is a stability matrix.

Condition (4.9) implies that for a sufficiently small period T, the spectral radius of $(\exp(\alpha_1 A_1 T) \exp(\alpha_2 A_2 T) \cdots \exp(\alpha_M A_M T))$ is less than one. The condition given in [20, Theorem 1] follows. For all practical purposes, this pulse-width modulation scheme results in an "average control" that asymptotically stabilizes system (4.1).

Unfortunately, finding the convex combination of (4.9), even when it exists, is NP-hard [99], [100]. Moreover, there is a large class of systems for which (4.9) is never satisfied, yet there exists a stabilizing $p(t)$. Example 3 with the A_i, $i = 1, 2$, given by (1.3a) is one such example: the system is stabilizable but no common P exists. General structural conditions on the A_i-matrices that guarantee stability remain open questions. Results in this direction can be found in [38] and [43]–[45]. The main thrust appears to be the existence of solutions to coupled Lyapunov equations as in [38], [43], and [45]. However, in the absence of structural conditions, the *min-projection strategy* in Section III provides a means for constructing a stabilizing switching strategy when appropriate P_i and Q_i are known. Application of the min-projection strategy to Example 3 results in a convergence rate much faster than illustrated in Fig. 3.

Related work using output feedback for stabilization can be found in Feron [36], Liberzon [46], and Savkin *et al.* [47].

V. Stability Conditions as Linear Matrix Inequalities

The critical challenge in practical hybrid system applications is finding appropriate Lyapunov functions that satisfy the stability conditions. Unfortunately, no general methods are available. However, for switched linear systems as in the previous sections, there is an LMI [79] problem formulation for constructing a set of quadratic Lyapunov-like functions [11], [80]–[86]. Existence of a solution to the LMI problem is a sufficient condition for hybrid system stability. A real advantage here is that LMI problems admit efficient and reliable numerical solutions with standard packages [87], [88].

For the case when one desires to guarantee stability for all switching sequences among asymptotically stable A_i-matrices, then the LMI problem is to find $P = P^T > 0$ satisfying $A_i^T P + P A_i < 0$, for $i = 1, \cdots, M$. It then follows that the set $\{A_1, \cdots, A_M\}$ is exponentially stable for all switching sequences $p(t)$ [80], [81], [83]. Conversely, if there exist positive definite matrices, R_i, $i = 1, \cdots, M$, such that

$$\sum_i \left(A_i^T R_i + R_i A_i\right) > 0 \tag{5.1}$$

then the above matrix P does not exist [80], [81]. Existence of $R_i > 0$ satisfying (5.1) means there exists a destabilizing switching sequence.

The general hybrid stability LMI formulation is much more complex. Broadly speaking, the problem entails searching for Lyapunov-like functions whose associated Ω-regions cover the state space. Specifically, the methodology begins with a sufficiently rich (user-chosen) partitioning of the state-space into potential Ω-regions. Each Ω-region must be defined by, or covered by, a region defined by a quadratic form. Presuming that each Ω-region is defined by a quadratic form, the search for appropriate Lyapunov-like functions is formulated as an LMI problem. Physical insight, a good understanding of the LMI problem, and brute force are often required to choose an acceptable partitioning. To generate a partitioning that admits a solution, we permit the use of several Lyapunov-like functions for each vector field.

Let Ω_q denote a region where one searches for a P_q in the quadratic Lyapunov-like function $V_q = x^T P_q x$ that satisfies the stability condition

$$\dot{V}_q(x) = \left[\frac{\partial}{\partial x} V_q(x)\right] A_i x = x^T \left(A_i^T P_q + P_q A_i^T\right) x \le 0 \tag{5.2}$$

with the understanding that several vector fields $A_i x$ can be used in Ω_q. Additionally, the LMI problem formulation requires that whenever there is movement to an adjacent region Ω_r, which uses the Lyapunov-like function V_r, then

$$V_r(x) \le V_q(x) \tag{5.3}$$

at those states x where the trajectory passes from Ω_q to Ω_r. Unlike the results set forth in Section III, this LMI problem formulation requires the stronger condition that the overall Lyapunov-like function must always be nonincreasing. This requirement necessitates the need for multiple (local) Lyapunov-like functions for each vector field. The concatenation of these Lyapunov-like functions, if they exist, produces a piecewise quadratic and piecewise differentiable (nontraditional) Lyapunov function.

The stability condition of (5.2) for a specific Lyapunov-like function must only be satisfied in the local region, for example, in Ω_q. To constrain the stability conditions to local regions, two steps are involved. First, the region must be expressed or contained in regions characterized by quadratic forms, $x^T Q x \ge 0$. Examples of such quadratic forms are cones and ellipsoids. If a polyhedral region is described by half-planes

$$c_a^T x \ge 0 \quad \text{and} \quad c_b^T x \ge 0$$

and

$$c_a^T x \le 0 \quad \text{and} \quad c_b^T x \le 0$$

then the quadratic form characterizing the region is obtained by multiplying the two half-planes together

$$x^T Q x \ge 0$$

where

$$Q = c_a c_b^T + c_b c_a^T. \tag{5.4}$$

In [49], [80]–[86], and [89], it is also shown how more general quadratic forms may be used to express hyperplanes and polyhedra.

Second, a technique called the S-procedure is applied to replace a constrained stability condition to a condition without constraints [79]. To illustrate, let A be a given matrix. The goal is to find $P > 0$ such that (5.2) is satisfied, i.e.,

$$x^T \left(A^T P + PA\right) x \le 0$$

for the region defined by $x^T Q x \ge 0$. Equations (5.2) and (5.4) represent the usual LMI problem for stability of a linear state model, except that we only require local validity here. By introducing a new unknown variable $\xi \ge 0$, there is potentially more freedom in finding a $P > 0$ and $\xi \ge 0$ satisfying the unconstrained relaxed condition

$$A^T P + PA + \xi Q \le 0 \tag{5.5}$$

intrinsic to the numerical (LMI) software [87]. A solution to the relaxed problem is also a solution to the constrained problem [79].

A solution to (5.5) insures that the system energy V is nonincreasing in the region $\{x | x^T Q x \ge 0\}$. Condition (5.3), that $V_r(x) \le V_q(x)$ at states x where the trajectory passes from Ω_q to Ω_r can, using local quadratic Lyapunov-like functions, be expressed by

$$x^T P_r x \le x^T P_q x. \tag{5.6}$$

The states where this condition must be satisfied also have to be expressed or contained in regions characterized by quadratic forms. As above, this new region-based constraint can be replaced with an unconstrained LMI condition using the S-procedure.

There is one final requirement: the Lyapunov-like function has to be positive definite, i.e., $x^T P_q x > 0$ locally, i.e., in Ω_q. This quadratic form condition can again be converted to an unconstrained condition using the S-procedure.

Instead of verifying stability, exponential stability can be shown by imposing slightly stronger conditions on the energy decrease. By finding upper and lower bounds on the Lyapunov-like functions, and lower bounds on the rate of energy decrease, exponential stability is guaranteed [49], [86]. Going through the steps described above, with the search for an upper bound of the convergence rate, exponential stability of hybrid systems with linear vector fields can be verified by the following LMI problem.

LMI Problem: Let I denote the identity matrix. If there is a solution to $\min \beta$ subject to:

1) $\alpha > 0,\ \mu_q \ge 0,\ \upsilon_q \ge 0,\ \xi_q \ge 0,$ $q \in \{1, \cdots, \kappa\}$;
2) $\alpha I + \mu_q Q_q \le P_q \le \beta I - \upsilon_q Q_q, \qquad q \in \{1, \cdots, \kappa\}$;
3) $A_i^T P_q + P_q A_i + \xi_q Q_q \le -I, \qquad (q, i) \in D_1$;
4) $P_r + \eta_{q,r} Q_{q,r} \le P_q, \qquad (q, r) \in D_2$;

where κ denotes the number of different local Lyapunov-like functions, then the equilibrium point 0 is exponentially stable in the sense of Lyapunov.

The left-hand side of the second condition requires positive definiteness of the quadratic Lyapunov-like functions. The right-hand side is introduced to find an upper bound of the Lyapunov-like functions to determine an upper bound of the convergence rate. If only stability is of interest, this right-hand side of the inequality can be neglected. The third condition is the requirement that the energy is nonincreasing in every region Ω_q, where D_1 is the set of tuples characterizing those vector fields allowable in region Ω_q. The energy decrease has to be less than the negative identity matrix to conclude exponential stability. Stability is guaranteed if the right-hand side instead is zero. Finally, the fourth condition is the requirement that the energy is nonincreasing when another region is entered, where D_2 is the set of tuples characterizing neighboring regions for which $x(t)$ can possibly travel from Ω_q to Ω_r.

Exponential stability is verified if there is a solution to the above LMI problem. The variables α, μ_q, υ_q, and ξ_q and matrices P_q are unknowns, while the different Q's are known matrices corresponding to the different user-generated local regions where the conditions must be valid. A bound on the convergence rate is

$$\|x(t)\| \le \sqrt{\frac{\beta}{\alpha}}\, e^{-(1/2\beta)t}\|x_0\| \tag{5.7}$$

where $x(t)$ is the continuous trajectory with initial state x_0. Because this bound depends on the user-chosen partitioning, one must exercise caution when interpreting this convergence rate; other switching sequences may result in much faster rates of convergence.

Example 6: The LMI method is illustrated for the system in Example 3. To verify exponential stability, the state space will be partitioned into two different regions: the first region corresponds to the states where the first vector field is used and the second region corresponds to the states where the second vector field is used, i.e., V_1 corresponds to A_1 and V_2 to A_2, as in Example 3.

The quadratic forms $x^TQ_1x \ge 0$ and $x^TQ_2x \ge 0$ restricting the Lyapunov-like functions V_1 and V_2 to the corresponding local regions are given by

$$Q_1 = \begin{bmatrix} -0.25 & -0.25 \\ -0.25 & 2 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 0.25 & 0.25 \\ 0.25 & -2 \end{bmatrix}$$

obtained according to (5.4) using the corresponding hyperplanes.

The trajectories pass the region boundaries in a clockwise direction, implying the formulation of the stability condition as $P_2 \le P_1$ at plane $x_2 = 0.5x_1$ and $P_1 \le P_2$ at $x_2 = -0.25x_1$. This is formulated as the third condition in the LMI problem by writing these planes as quadratic forms and using the S-procedure.

Solving the LMI problem results in a solution

$$P_1 = \begin{bmatrix} 0.1000 & -0.4500 \\ -0.4500 & 41.1167 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 4.3792 & 3.8292 \\ 3.8292 & 6.8833 \end{bmatrix}$$

with a value of $\beta = 41.12$. Hence, the system is (globally) exponentially stable. The level curves of the Lyapunov-like functions are shown in Fig. 7. Note that in this example, the resulting Lyapunov-like function is always nonincreasing; see Fig. 4. □

Various extensions of the LMI problem are possible. A generalization to affine systems, $\dot{x} = A_ix + B_i$, utilizes a general quadratic Lyapunov function and more general quadratic forms to handle ellipsoid regions with nonzero centers and hyperplanes with offsets [49], [79], [89]. In some instances, the LMI problem can be adapted to nonlinear vector fields [49], [84], [86], [89]. Incorporation of state jumps is also possible [49]. More recently, LMI methods have been extended to synthesis for example in the case of piecewise linear systems [90]. Stabilization in the context of linear programming and optimal control is considered in [101].

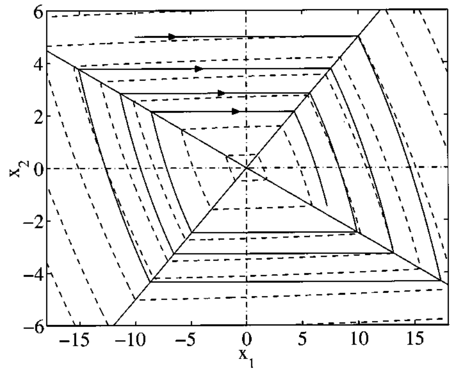

Fig. 7. Dashed line (– –) shows the level curves of the local quadratic Lyapunov-like functions. Continuous trajectories cross these curves such that the energy is always decreasing, thus verifying that the system is exponentially stable.

VI. Stability Analysis: Two Examples

Example 4 (Continued): The flight control problem of the longitudinal dynamics of an aircraft set forth in example 4 used the max control law of (2.3) and (2.4)

$$u = \max(-Fx + k_1\alpha_{\rm lim},\, -Gx + k_2r(t)).$$

Herein, we examine the stability of the closed-loop system when the tracking command $r(t) \equiv 0$. See [51] and [59] for more details. The closed-loop system equations with the max control law are

$$\begin{aligned}\dot{x} &= Ax + B\max(-Fx + k_1\alpha_{\rm lim},\, -Gx) \\ &= (A - BG)x + B\max((G - F)x + k_1\alpha_{\rm lim},\, 0).\end{aligned}$$

Our analysis below presupposes that the feedback controls F and G are designed so that $(A\text{–}BF)$ and $(A\text{–}BG)$ are stable and provide the necessary performance. This is possible because the controllability of (A, B) is equivalent to the ability to reassign the eigenvalues of A by state feedback. (Achieving desired performance using hybrid controllers is a topic more complex than stability alone and is explored in by McClamroch and Kolmanovsky in [102].) After defining $\gamma = k_1\alpha_{\rm lim}$ and making a change of variables, one obtains the following "canonical form" [51], [59]:

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -\alpha_1 z_1 - \alpha_2 z_2 + \max(\alpha_3 z_1 + \alpha_4 z_2 + \gamma,\, 0) \end{bmatrix} \tag{6.1}$$

where α_1, α_2, α_1–α_3, and α_2–α_4 are greater than zero. Further, without loss of generality, we assume that $\gamma \le 0$, in which case the only equilibrium point of this system is the origin. Global asymptotic stability of the equilibrium dynamics is guaranteed if for an appropriate Lyapunov function $V(z)$ a) the origin is the only invariant set for which $\dot{V} = 0$ and b) $V(z) \to \infty$ as $\|z\| \to \infty$ [103].

Hitherto, we have illustrated the results with quadratic Lyapunov functions. To verify stability of this example, we claim that the noncustomary

$$\begin{aligned} V &= \tfrac{1}{2}\alpha_3 \dot{z}_1^2 + \int_0^{z_1} [\alpha_1 \xi - \max(\alpha_3 \xi + \gamma, 0)]\, d\xi \\ &\equiv \tfrac{1}{2}\alpha_3 \dot{x}^2 + \int_0^{x} c(\xi)\, d\xi \end{aligned}$$

is a valid Lyapunov function for the system of (6.1).

The verification has two major parts: a) V is a positive definite (p.d.) function and b) $\dot{V} \le 0$. To show that V is p.d., it suffices to show that $c(0) = 0$ and $z_1 c(z_1) > 0$ when $z_1 \ne 0$. That $c(0) = 0$ follows from $\gamma \le 0$. Second,

$$\begin{aligned} z_1 c(z_1) &= a x^2 - z_1 \max(f z_1 + \gamma, 0) \\ &= \begin{cases} \alpha_1 z_1^2 > 0, & \alpha_3 z_1 + \gamma \le 0 \\ \alpha_1 z_1^2 - \alpha_3 z_1^2 - \gamma z_1 > 0, & \alpha_3 z_1 + \gamma > 0. \end{cases} \end{aligned}$$

That the desired condition holds in the first case follows immediately from $\alpha_1 > 0$. For the second case, we consider the two subcases $z_1 > 0$ and $z_1 < 0$ separately, which lead directly to the desired result. Thus, V is a p.d. function.

To verify that $\dot{V} \le 0$, consider

$$\begin{aligned} \dot{V} &= \dot{z}_1 \ddot{z}_1 + c(z_1)\dot{z}_1 \\ &= \dot{z}_1 [-\alpha_1 z_1 - \alpha_2 \dot{z}_1 + \max(\alpha_3 z_1 + \alpha_4 \dot{z}_1 + \gamma, 0)] \\ &\quad + \alpha_1 z_1 \dot{z}_1 - \max(\alpha_3 z_1 + \gamma, 0)\, \dot{z}_1 \\ &= -\alpha_2 \dot{z}_1^2 + \dot{z}_1 \max(\alpha_3 z_1 + \alpha_4 \dot{z}_1 + \gamma, 0) \\ &\quad - \dot{z}_1 \max(\alpha_3 z_1 + \gamma, 0). \end{aligned}$$

This last equality requires consideration of four cases: $\alpha_3 z_1 + \alpha_4 \dot{z}_1 + \gamma$.

1) If $\alpha_3 z_1 + \alpha_4 \dot{z}_1 + \gamma \le 0$ and $\alpha_3 z_1 + \gamma \le 0$, then $\dot{V} = -\alpha_2 \dot{z}_1^2 \le 0$.
2) If $\alpha_3 z_1 + \alpha_4 \dot{z}_1 + \gamma > 0$ and $\alpha_3 z_1 + \gamma > 0$, then $\dot{V} = -(\alpha_2 - \alpha_4)\dot{z}_1^2 \le 0$.
3) If $\alpha_3 z_1 + \alpha_4 \dot{z}_1 + \gamma \le 0$ and $\alpha_3 z_1 + \gamma > 0$, then $\dot{V} = -\alpha_2 \dot{z}_1^2 - \dot{z}_1(\alpha_3 z_1 + \gamma)$ is less than or equal to zero if $\dot{z}_1 \ge 0$ and is less than zero if $\dot{z}_1 < 0$ since $(\alpha_2 - \alpha_4) > 0$.
4) If $\alpha_3 z_1 + \alpha_4 \dot{z}_1 + \gamma > 0$ and $\alpha_3 z_1 + \gamma \le 0$, then $\dot{V} = -\alpha_2 \dot{z}_1^2 + \dot{z}_1(\alpha_3 z_1 + \alpha_4 \dot{z}_1 + \gamma)$ is less than or equal to zero if $\dot{z}_1 \le 0$ and is less than zero if $\dot{z}_1 > 0$ is.

Example 5 (Continued): Stability of the PI controlled vehicle with an automatic transmission given in Example 5 is verified by LMI's. By denoting $\Delta v = v_{\mathrm{ref}} - v$ and $\Delta T_I = T_I$ (T_I is the integral part of the control signal, the desired torque generated by the engine), the closed-loop dynamics in (2.8) becomes

$$\begin{bmatrix} \Delta \dot{v} \\ \Delta \dot{T}_I \end{bmatrix} = \begin{bmatrix} -G_{p(t)} K_{p(t)}/m & -G_{p(t)}/m \\ K_{p(t)}/T_R & 0 \end{bmatrix} \begin{bmatrix} \Delta v \\ \Delta T_I \end{bmatrix}$$

where $m = 1500$, $T_R = 40$, $G_{p(t)} \in \{50, 32, 20, 14\}$, $K_{p(t)} \in \{3.75, 5.86, 9.37, 13.39\}$, and $G_{p(t)} K_{p(t)} = 187.5$ for all discrete states. For a specified desired velocity v_{ref}(=30 m/s), the system converges exponentially to v_{ref}, illustrated in Fig. 8 and formally proven next.

Initially assuming no state jumps in T_I, stability can be shown by a single partitioning, implying a single Lyapunov-like function common to all the discrete states. This results in a solution

$$P = \begin{bmatrix} 255.589 & 72.262 \\ 72.262 & 40.822 \end{bmatrix}$$

satisfying the conditions of the LMI problem in Section V. Hence, the hybrid system is globally exponentially stable without state jumps. In (5.7), the optimal value of $\beta = 277.64$.

If the state jumps are included in the dynamics, they occur when the discrete state is changed. Trajectories satisfying the condition $T_I > K_{p(t)}(v - v_{\mathrm{ref}})$ cross the gear shifting lines $v = 1/G_{p(t)}\omega_{\mathrm{low}}$ and $v = 1/G_{p(t)}\omega_{\mathrm{high}}$ from left to right (see Fig. 7) and oppositely for $T_I < K_{p(t)}(v - v_{\mathrm{ref}})$. In the operating region of this cruise controller [$T_I(0)$ is always put to zero when a new desired velocity is given], the gear shiftings will always occur from lower to higher gear when the first condition is satisfied and conversely in the second case. Hence, the third condition of the LMI problem is formulated such that the energy decreases passing from gears i to $i+1$ satisfying $T_I > K_{p(t)}(v - v_{\mathrm{ref}})$ and gears $i+1$ to i satisfying $T_I < K_{p(t)}(v - v_{\mathrm{ref}})$.

Consider the case when the trajectories start in the first region. The jump condition (2.9b), becomes

$$F_i = \begin{bmatrix} 1 & 0 \\ 0 & \dfrac{G_{i+1}}{G_i} \end{bmatrix} \qquad i = 1, 2, 3.$$

Accordingly, the fourth LMI condition becomes

$$F_i^T P F_i \le P$$

which is equivalent to

$$p_{2,2} \left(\frac{G_{i+1}}{G_i} \right)^2 \le p_{2,2}$$

where $p_{2,2}$ is the (2, 2) element of P. Since $(G_{i+1}/G_i)^2 < 1$, the energy will decrease due to the state jumps for any quadratic function $x^T P x$. Therefore, the same solution as above verifies stability.

However, when the trajectories start in the second region, the jump condition of (2.9b) is the same as above except that G_i and G_{i+1} change position. In this case, there will not exist any solution since $(G_{i+1}/G_i)^2 > 1$.

To overcome this problem and verify stability, the state space is further partitioned. One quadratic candidate Lyapunov-like function is associated with each of the discrete states. Solving the LMI problem leads to a solution

$$P_1 = \begin{bmatrix} 304.082 & 87.089 \\ 87.089 & 376.934 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 248.013 & 79.625 \\ 79.625 & 144.215 \end{bmatrix}$$
$$P_3 = \begin{bmatrix} 212.101 & 59.328 \\ 59.328 & 53.788 \end{bmatrix}, \quad P_4 = \begin{bmatrix} 147.495 & 53.571 \\ 53.571 & 24.112 \end{bmatrix}.$$

Hence, the hybrid system is exponentially stable also in the case of state jumps. In (5.7), the optimal value of $\beta = 439.9$.

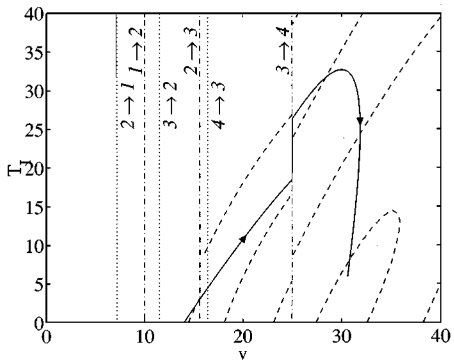

Fig. 8. Phase plot of v and T_I during a simulation of 80 s. Dash-dotted lines (– ·) are the hyperplanes when the gear shifts from lower to higher gears ($i \to i+1$) and oppositely ($i+1 \to i$) at the dotted lines (· ·). Dashed lines (– –) are the level curves of the local quadratic Lyapunov-like functions. The continuous trajectories cross these curves such that the energy decreases all the time, verifying that the system is exponentially stable.

The level curves for the local quadratic Lyapunov functions are shown in Fig. 8.

VII. Concluding Remarks

Control theorists and computer scientists are actively investigating all dimensions of hybrid systems, and hybrid system stability remains a vibrant topic within this broad community, as can be garnered from the large number of references below. Within the context of this widespread activity, this paper has surveyed results on the stability analysis of hybrid systems concentrating on the use of "multiple Lyapunov functions" as extensions of the classical theory [91]–[93]. We presented general theorems for nonlinear hybrid systems, stronger results for the case of switching among linear systems, and computer tools (LMI's) for the verification of hybrid systems stability. In the latter case, the existence of a solution to an LMI problem becomes a sufficient condition for the existence and construction of a stabilizing switching sequence for linear vector fields. Two extended case studies, along with a number of examples scattered throughout this paper, illustrated the main points and potential applications of the theory.

Despite the variety and significance of the many results on hybrid system stability, general necessary and sufficient conditions in terms of the structure of the vector fields, $\{f_i | i = 1, \cdots, M\}$ in the general case and $\{A_i | i = 1, \cdots, M\}$ in the switched linear case, have evaded discovery. By structural stability results, we mean conditions similar to those of classical linear state space theory where BIBS and BIBO stability [113] are characterized by the eigenvalue locations of the A-matrix and the system controllability and observability. What properties of the vector fields $\{f_i\}$ or $\{A_i\}$ lead to the existence and "straightforward" construction of stabilizing switching sequences?

Another foundational approach toward the study of hybrid system stability is the work of [2], [3], [8], and [15]. This work introduces an extended notion of time (of which real time is a projection) and the notion of a stability preserving map. The classical and MLF approach are two types of stability preserving maps. Other maps are possible, and it would seem somewhat narrow to think that Lyapunov-like maps are sufficient to characterize stability of the variety of motions exhibited by hybrid systems that can be chaotic [104].

The problem of asymptotic stabilization of C/DT systems via quantized feedback is also a problem of recent interest [48], [105]–[107]. The quantization condition makes this a hybrid system problem. Of particular note are the cases where the control engineer can 1) specify the quantization scheme and/or 2) adaptively change the quantization levels as a function of time.

The insights afforded by hybrid system stability investigations have recently led to a revisitation of the problem of nonlinear sampled-data feedback [6], [15], [21]–[23], [108]. These results are important contributions and can be nicely viewed as part of the larger topic of looking at stability of first (i.e., linear) approximations extended to the case of hybrid systems as in [6] with consequences for the stability of feedback loops closed over asynchronous networks [109].

There are many other issues related to hybrid system stability and stabilization. For example, besides the need for structural stability conditions, there are computational challenges related to overcoming the complexity of constructing stabilizing switching sequences. See, for example, [99], [100], and [110]–[112]. More important, there is a need to apply and extend the basic results surveyed in this paper to the development of systematic analysis and design tools for practical hybrid systems. Important application areas such as automotive controls, avionics, and embedded controllers in general will drive the process. Indeed, we have just stepped into the arena where we must confront and deal with the intrinsically hybrid nature of virtually all complex systems.

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References

General Stability Theory

[1] M. S. Branicky, "Multiple Lyapunov functions and other analysis tools for switched and hybrid systems," *IEEE Trans. Automat. Contr.*, vol. 43, no. 4, pp. 475–482, 1998.

[2] H. Ye, A. N. Michel, and L. Hou, "Stability theory for hybrid dynamical systems," *IEEE Trans. Automat. Contr.*, vol. 43, no. 4, pp. 461–474, Apr. 1998.

[3] A. N. Michel and B. Hu, "Toward a stability theory of general hybrid dynamical systems," *Automatica*, vol. 35, pp. 371–384, 1999.

[4] A. S. Morse, *Control Using Logic-Based Switching*. London, U.K.: Springer-Verlag, 1997.

[5] P. Peleties and R. A. DeCarlo, "Asymptotic stability of m-switched systems using Lyapunov-like functions," in *Proc. 1991 American Control Conf.*, Boston, MA, 1991, pp. 1679–1684.

[6] M. S. Branicky, "Stability of switched and hybrid systems: State of the art," in *Proc. 36th IEEE Conf. Decision and Control*, 1997, pp. 1208–1212.

[7] R. Alur, T. A. Henzinger, and E. D. Sontag, Eds., *Hybrid Systems III: Verification and Control*. Berlin, Germany: Springer-Verlag, 1996, Lecture Notes in Computer Science.

[8] A. N. Michel, "Recent trends in the stability analysis of hybrid dynamical systems," *IEEE Trans. Circuits Syst. I*, vol. 46, no. 1, pp. 120–134, Jan. 1999.

[9] W. P. Dayawansa and C. F. Martin, "A converse Lyapunov theorem for a class of dynamical systems which undergo switching," *IEEE Trans. Automat. Contr.*, vol. 44, pp. 751–760, 1999.
[10] M. S. Branicky, "Stability of switched and hybrid systems," in *Proc. 33rd IEEE Conf. Decision and Control*, Lake Buena Vista, FL, Dec. 14–16, 1994, pp. 3498–3503.
[11] S. Pettersson and B. Lennartson, "Stability and robustness for hybrid systems," in *Proc. 35th IEEE Conf. Decision and Control*, Kobe, Japan, 1996, pp. 1202–1207.
[12] ——, *Controller Design of Hybrid Systems*. ser. Lecture Notes in Computer Science 1201, Oded Maler, Ed. Berlin, Germany: Springer, 1997, pp. 240–254.
[13] Z. Artstein, "Examples of stabilization with hybrid feedback," in *Hybrid Systems III*, R. Alur, T. A. Henzinger, and E. D. Sontag, Eds. New York: Springer, 1996, pp. 173–185.
[14] M. Dogruel and U. Ozguner, "Stability of hybrid systems," in *Proc. IEEE Int. Symp. Intelligent Control*, 1994, pp. 129–134.
[15] H. Ye, A. N. Michel, and L. Hou, "Stability theory for hybrid dynamical systems," in *Proc. 34th IEEE Conf. Decision and Control*, New Orleans, LA, 1995, pp. 2679–2684.
[16] B. Hu and A. N. Michel, "Some qualitative properties of multirate digital control systems," *IEEE Trans. Automat. Contr.*, vol. 44, Apr. 1999.
[17] H. Ye, A. N. Michel, and L. Hou, "Stability analysis of systems with impulse effects," *IEEE Trans. Automat. Contr.*, vol. 43, pp. 1719–1723, Dec. 1998.
[18] T. Pavlidis, "Stability of systems described by differential equations containing impulses," *IEEE Trans. Automat. Contr.*, vol. AC-12, pp. 43–45, Feb. 1967.
[19] D. D. Bainov and P. S. Simeonov, *Systems with Impulse Effect*. Chichester, U.K.: Ellis Horwood, 1989.
[20] J. Ezzine and A. H. Haddad, "Controllability and observability of hybrid systems," *Int. J. Control*, vol. 49, no. 6, pp. 2045–2055, June 1989.
[21] J. P. Hespanha and A. S. Morse, "Stabilization of nonholonomic integrators via logic-based switching," *Automatica*, vol. 35, pp. 385–393, 1999.
[22] A. T. Barabanov and Y. F. Starozhilov, "Investigation of the stability of continuous discrete systems by Lyapunov's second method," *Sov. J. Autom. Inf. Sci.*, vol. 21, no. 6, pp. 35–41, Nov.–Dec. 1988.
[23] L. Hou, A. N. Michel, and Y. Hui, "Some qualitative properties of sampled-data control systems," in *Proc. 35th IEEE Conf. Decision and Control*, Kobe, Japan, Dec. 1996, pp. 911–916.

Switched System Stability

[24] T. I. Seidman, "Switching systems," UMBC, Baltimore, MD, Tech. Rep. 86-78, 1986.
[25] ——, "Optimal control for switching systems," in *21st Annu. Conf. Information Science and Systems*, Johns Hopkins University, 1987, pp. 485–489.
[26] D. Liberzon and A. S. Morse, "Basic problems in stability and design of switched systems," *IEEE Contr. Syst. Mag.*, vol. 19, no. 5, pp. 59–70, Oct. 1999.
[27] T. Ooba and Y. Funahashi, "Two conditions concerning common quadratic Lyapunov functions for linear systems," *IEEE Trans. Automat. Contr.*, vol. 42, pp. 719–721, 1997.
[28] R. N. Shorten and K. S. Narendra, "On the stability and existence of common Lyapunov functions for stable linear switching systems," in *Proc. 37th IEEE Conf. Decision and Control*, 1998, pp. 3723–3724.
[29] A. A. Agrachev and D. Liberzon, "Lie-algebraic conditions for exponential stability of switched systems," in *Proc. of the 38th Conf. Decision and Control*, Phoenix, AZ, Dec. 1999, pp. 2679–2684.
[30] M. Wicks, P. Peleties, and R. DeCarlo, "Switched controller synthesis for the quadratic stabilization of a pair of unstable linear systems," *Eur. J. Contr.*, vol. 4, pp. 140–147, 1998.
[31] K. A. Loparo, J. T. Aslanis, and O. Hajek, "Analysis of switching linear systems in the plane, Part I: Local behavior of trajectories and local cycle geometry," *J. Optimiz. Theory Applicat.*, vol. 52, no. 3, pp. 365–394, 1987.
[32] ——, "Analysis of switching linear systems in the plane, Part II: Global behavior of trajectories, controllability, and attainability," *J. Optimiz. Theory Applicat.*, vol. 52, no. 3, pp. 395–427, 1987.
[33] X. Xu and P. Antsaklis, "Design of stabilizing control laws for second-order switched systems," in *Proc. 14th IFAC World Congress*, vol. C, Beijing, P.R.C., 1999, pp. 181–186.
[34] B. Hu, X. Xu, P. J. Antsaklis, and A. N. Michel, "Robust stabilizing control law for a class of second-order switched systems," *Syst. Contr. Lett.*, vol. 38, pp. 197–207, 1999.
[35] M. A. Wicks, P. Peleties, and R. A. DeCarlo, "Construction of piecewise Lyapunov functions for stabilizing switched systems," in *Proc. 33rd IEEE Conf. Decision and Control*, Lake Buena Vista, FL, Dec. 14–16, 1994, pp. 3492–3497.
[36] E. Feron, "Quadratic stabilizability of switched systems via state and output feedback," Center for Intelligent Control Systems, Massachusetts Institute of Technology, Tech. Rep. CICS P-468, Feb. 1996.
[37] B. E. Bishop, "Lyapunov function normalization for controlled switching of hybrid systems," in *Proc. 1999 Amer. Control Conf.*, San Diego, CA, Dec. 1999.
[38] J. Malmborg, B. Bernhardsson, and K. J. Astrom, "A stabilizing switching scheme for multi-controller systems," in *Proc. IFAC World Congress*, San Francisco, CA, July 1996.
[39] K. S. Narendra and J. Balakrishnan, "A common Lyapunov function for stable LTI systems with commuting A-matrices," *IEEE Trans. Automat. Contr.*, pp. 2469–2471, 1994.
[40] D. Liberzon, J. P. Hespanha, and A. S. Morse, "Stability of switched systems: A Lie-algebraic condition," *Syst. Contr. Lett.*, vol. 37, no. 3, pp. 117–122, 1999.
[41] L. Gurvits, "Stability of discrete linear inclusions," *Lin. Alg. Appl.*, vol. 231, pp. 47–85, 1995.
[42] J. Hespanha and A. S. Morse, "Stability of switched systems with average dwell-time," in *Proc. 38th Conf. Decision and Control*, Phoenix, AZ, Dec. 1999, pp. 2655–2660.
[43] M. A. Wicks and R. A. DeCarlo, "Solution of coupled Lyapunov equations for the stabilization of multimodal linear systems," in *Proc. 1997 Amer. Control Conf.*, Albuquerque, NM, June 1997, pp. 1709–1713.
[44] A. Y. Pogromsky, M. Jirstrand, and P. Spangéus, "On stability and passivity of a class of hybrid systems," in *Proc. 37th IEEE Conf. Decision and Control*, San Diego, CA, 1998, pp. 3705–3710.
[45] L. Xie, S. Shishkin, and M. Fu, "Piecewise Lyapunov functions for robust stability of linear time-varying systems," *Syst. Contr. Lett.*, vol. 31, pp. 165–171, 1997.
[46] D. Liberzon, "Stabilizing a linear system with finite-state hybrid output feedback," in *Proc. 7th IEEE Mediterranean Conf. Control and Automation*, Haifa, Israel, 1999, pp. 176–183.
[47] A. V. Savkin, E. Skafidas, and R. J. Evans, "Robust output feedback stabilizability via controller switching," *Automatica*, vol. 35, pp. 69–74, 1999.
[48] R. W. Brockett and D. Liberzon, "Quantized feedback stabilization of linear systems," *IEEE Trans. Automat. Contr.*, July 2000, to be published.

Dissertations

[49] S. Pettersson, "Analysis and design of hybrid systems," Ph.D. dissertation, Chalmers Univ. of Technology, Goteborg, Sweden, June 1999.
[50] P. A. Peleties, "Modeling and design of interacting continuous-time/discrete event systems," Ph.D. dissertation, School of Electrical Engineering, Purdue Univ., West Lafayette, IN, Dec. 1992.
[51] M. S. Branicky, "Studies in hybrid systems: Modeling, analysis, and control," Ph.D. dissertation, Dept. of Electrical and Computer Engineering, Massachusetts Institute of Technology, 1995.
[52] H. Ye, "Stability analysis of two classes of dynamical systems: General hybrid dynamical systems and neural networks with delays," Ph.D. dissertation, Dept. of Electrical Engineering, Univ. of Notre Dame, May 1996.
[53] K. H. Johansson, "Relay, feedback, and multivariable control," Ph.D. dissertation, Dept. of Automatic Control, Lund Institute of Technology, Lund, Sweden, May 1998.
[54] J. Malmborg, "Analysis and design of hybrid control systems," Ph.D. dissertation, Dept. of Automatic Control, Lund Institute of Technology, Lund, Sweden, May 1998.
[55] J. P. Hespanha, "Logic-based switching algorithms in control," Ph.D. dissertation, Dept. of Electrical Engineering, Yale Univ., 1998.

Applications

[56] A. Gollu and P. O. Varaiya, "Hybrid dynamical systems," in *Proc. 28th IEEE Conf. Decision and Control*, Tampa, FL, Dec. 1989, pp. 2708–2712.

[57] R. W. Brockett, "Hybrid models for motion control systems," in *Essays on Control: Perspectives in the Theory and its Applications*, H. L. Trentelman and J. C. Willems, Eds. Boston, MA: Birkhauser, 1993, pp. 29–53.

[58] M. Oishi and C. Tomlin, "Switched nonlinear control of a VSTOL aircraft," in *Proc. 38th Conf. Decision and Control*, Phoenix, AZ, Dec. 1999, pp. 2685–2690.

[59] M. S. Branicky, "Analysis of continuous switching systems: Theory and examples," in *Proc. Amer. Control Conf.*, Baltimore, MD, June 29–July 1, 1994, pp. 3110–3114.

[60] H. Sira-Ramirez, "Nonlinear P-I controller design for switched mode dc-to-dc power converters," *IEEE Trans. Circuits Syst.*, vol. 38, no. 4, pp. 410–417, Apr. 1991.

[61] S. R. Sanders and G. C. Verghese, "Lyapunov-based control for switched power converters," in *Proc. 21st Annu. IEEE Power Electronics Specialists Conf. (PESC'90)*, San Antonio, TX, June 11–14, 1990, Part 1 (of 2).

[62] G. Szentirmai and G. C. Temes, "Switched-capacitor building blocks," *IEEE Trans. Circuits Syst.*, vol. CAS-27, June 1980.

Modeling

[63] H. S. Witsenhausen, "A class of hybrid-state continuous-time dynamic systems," *IEEE Trans. Automat. Contr.*, vol. AC-11, no. 2, pp. 161–167, 1966.

[64] L. Tavernini, "Differential automata and their discrete simulators," *Nonlinear Anal., Theory, Methods, Appl.*, vol. 11, no. 6, pp. 665–683, 1987.

[65] P. Peleties and R. DeCarlo, "Modeling of interacting continuous time and discrete event systems: An example," in *Proc. 26th Annu. Allerton Conf.*, Sept. 1988, pp. 1150–1159.

[66] ——, "A modeling strategy for discrete event supervisory control of continuous time systems," in *Proc. 28th IEEE Conf. Decision and Control*, Dec. 1989, pp. 1308–1313.

[67] ——, "A modeling strategy for hybrid systems based on event structures," *Discrete Event Dynamic Syst.*, vol. 3, pp. 39–69, 1993.

[68] ——, "Analysis of a hybrid system using symbolic dynamics and petri nets," *Automatica*, vol. 30, no. 9, pp. 1421–1427, Sept. 1994.

[69] J. A. Stiver, P. J. Antsaklis, and M. D. Lemmon, "A logical DES approach to the design of hybrid control systems," *Mathematical and Computer Modeling*, vol. 23, no. 11/12, pp. 55–76, June 1996. (Special Issue on Discrete Event Systems).

[70] P. J. Antsaklis, J. A. Stiver, and M. D. Lemmon, "Hybrid system modeling and autonomous control systems," in *Hybrid Systems*. New York: Springer-Verlag, 1993, pp. 366–392.

[71] R. Alur, C. Courcoubetis, T. A. Henzinger, and P.-H. Ho, Eds., "Hybrid automata: An algorithmic approach to the specification and verification of hybrid systems," in *Workshop on Theory of Hybrid Systems.* Lyngby, Denmark, 1992, vol. 736, Lecture Notes in Computer Science, pp. 209–229.

[72] R. Grossman, A. Nerode, A. Ravn, and H. Rischel, *Hybrid Systems*, H. Rischel, Ed. New York: Springer-Verlag, 1993, vol. 736, Lecture Notes in Computer Science.

[73] A. Nerode and W. Kohn, "Models for hybrid systems: Automata, topologies, controllability, observability," in *Hybrid Systems*. New York: Springer-Verlag, 1993, pp. 317–350.

[74] M. Dogruel and U. Ozguner, "Stability of hybrid systems," in *Proc. IEEE Int. Symp. Intelligent Control*, 1994, pp. 129–134.

[75] M. S. Branicky, V. S. Borkar, and S. K. Mitter, "A unified framework for hybrid control: Model and optimal control theory," *IEEE Trans. Automat. Contr.*, vol. 43, no. 1, pp. 31–45, 1998.

[76] ——, "A unified framework for hybrid control," in *Proc. 33rd IEEE Conf. Decision and Control*, Lake Buena Vista, FL, Dec. 1994, pp. 4228–4234.

[77] S. Pettersson and B. Lennartson, "Hybrid modeling focused on hybrid petri nets," in *Proc. 2nd Eur. Workshop Real-Time and Hybrid Systems*, Grenoble, France, June 1995, pp. 303–309.

[78] B. Lennartson, M. Tittus, B. Egardt, and S. Pettersson, "Hybrid systems in process control," *IEEE Contr. Syst. Mag.*, vol. 16, no. 5, pp. 45–55, 1996.

LMI Applications

[79] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*. Philadelphia, PA: SIAM, 1994.

[80] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," *IEEE Trans. Automat. Contr.*, vol. 43, no. 4, pp. 555–559, 1998.

[81] ——, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," Dept. of Automatic Control, Lund Institute of Technology, Tech. Rep. ISRN LUTFD2/TFRT-7549-SE, June 1996.

[82] ——, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," in *Proc. 4th Eur. Contr. Conf.*, Brussels, Belgium, 1997.

[83] S. Pettersson and B. Lennartson, "Stability and robustness for hybrid systems," Control Engineering Laboratory, Chalmers Univ. of Technology, Tech. Rep. I-96/003, July 1996.

[84] ——, "LMI for stability and robustness of hybrid systems," in *Proc. Amer. Control Conf.*, Albuquerque, NM, 1997, pp. 1714–1718.

[85] ——, "Exponential stability analysis of nonlinear systems using LMIs," in *Proc. 36th IEEE Conf. Decision and Control*, San Diego, CA, 1997, pp. 199–204.

[86] ——, "Exponential stability of hybrid systems using piecewise quadratic Lyapunov functions resulting in an LMI problem," in *Proc. 14th IFAC World Congress*, Beijing, China, July 1999.

[87] P. Gahinet, A. Nemirovski, A. J. Laub, and M. Chilali, *LMI Control Toolbox for Use with Matlab*: Mathworks, May 1995.

[88] L. El Ghaoui, *LMITOOL: An Interface to Solve LMI Problems*: ENSTA, 1995.

[89] M. Johansson, "Piecewise linear control systems," Lund Institute of Technology, Lund, Sweden, ISRN LUTFD2/TFRT-1050-SE, 1999.

[90] A. Rantzer and M. Johansson, "Piecewise linear quadratic optimal control," *IEEE Trans. Automat. Contr.*, to be published.

Miscellaneous

[91] V. I. Zubov, *Methods of A. M. Lyapunov and Their Applications*. Gronigen, The Netherlands: Noordhoff, 1964.

[92] W. Hahn, *Stability of Motion*. Berlin and New York: Springer-Verlag, 1967.

[93] T. Yoshizawa, *Stability Theory by Lyapunov's Second Method*. Tokyo, Japan: Math. Soc. Japan.

[94] A. F. Filippov, *Differential Equations with Discontinuous Right Hand Sides*. Norwell, MA: Kluwer Academic, 1988.

[95] D. Shevitz and B. Paden, "Lyapunov stability of nonsmooth systems," *IEEE Trans. Automat. Contr.*, vol. 39, no. 9, pp. 181–191, 1994.

[96] V. I. Utkin, "Sliding mode control in discrete-time and difference systems," in *Variable Structure and Lyapunov Control*, A. S. Zinoear, Ed. London, U.K.: Springer-Verlag, 1993, pp. 87–107.

[97] K. S. Narendra and A. M. Annaswamy, *Stable Adaptive Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1989.

[98] A. D. Molchanov and Y. S. Pyatnitsky, "Criteria of absolute stability of differential and difference inclusions encountered in control theory," *Syst. Contr. Lett.*, vol. 13, pp. 59–64, 1989.

[99] V. Blondel and J. Tsitsiklis, "NP-hardness of some linear control design problems," *SIAM J. Contr. Optimiz.*, vol. 35, no. 6, pp. 2118–2127, 1997.

[100] V. D. Blondel and J. N. Tsitsiklis, "Complexity of stability and controllability of elementary hybrid systems," *Automatica*, vol. 35, no. 3, pp. 479–489, Mar. 1999.

[101] S. Hedlund and A. Rantzer, "Optimal Control of Hybrid Systems," in *Proc. 1999 IEEE Conf. Decision and Control*, Phoenix, AZ, Dec. 1999.

[102] N. H. McClamroch and I. Kolmanovsky, "Performance benefits of hybrid control design for linear and nonlinear systems," in *Proceedings of the IEEE*, vol. 88, July 2000, pp. XXX–XXX.

[103] J.-J. Slotine and W. Li, *Applied Nonlinear Control*. Upper Saddle River, NJ: Prentice-Hall, 1991.

[104] C. Chase, J. Serrano, and P. J. Ramadge, "Periodicity and chaos from switched flow systems: Contrasting examples of discretely controlled continuous systems," *IEEE Trans. Automat. Contr.*, vol. 38, no. 1, pp. 70–83, 1993.

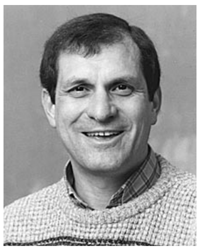
[105] N. Elia, "Quantized linear systems," in *Systems Theory: Modeling, Analysis, and Control*, T. E. Djaferis and I. C. Schick, Eds. Boston, MA: Kluwer Academic, 1999, pp. 433–462.

[106] B. Hu, Z. Feng, and A. N. Michel, "Quantized sampled-data feedback stabilization forlinear and nonlinear control systems," in *Proc. 38th IEEE Conf. Decision and Control*, Phoenix, AZ, 1999, pp. 4392–4397.

[107] D. Liberzon, "Nonlinear stabilization by hybrid quantized feedback," in *Proc. 3rd Int. Workshop Hybrid Systems: Computation and Control*, Pittsburgh, PA, Mar. 2000, to be published.

[108] C. Rui, I. Kolmanovsky, and N. H. McClamroch, "Hybrid control for stabilization of a class of cascade nonlinear systems," in *Proc. American Control Conf.*, Albuquerque, NM, 1997, pp. 2800–2804.

[109] M. S. Branicky, S. P. Phillips, and W. Zhang, "Stability of networked control systems: Explicit analysis of delay," in *Proc. Amer. Control Conf.*, Chicago, IL.
[110] L. Gurvits and L. Rodman, "Convergence of polynomially bounded semigroups of matrices," *SIAM J. Matrix Anal. Applicat.*, vol. 18, no. 2, pp. 360–368, 1997.
[111] O. Toker, "On the algorithmic unsolvability of some stability problems for discrete event systems," in *Proc. IFAC World Congress*, San Francisco, CA, 1996, pp. 353–358.
[112] M. S. Branicky, "Universal computation and other capabilities of hybrid and continuous dynamical systems," *Theoret. Comput. Sci.*, vol. 138, no. 1, pp. 67–100, 1995.
[113] W. J. Rugh, *Linear System Theory*, 2nd ed. Englewood Cliffs, NJ: Prentice-Hall, 1996.



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