

# Stability of Networked Control Systems: Explicit Analysis of Delay

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## Abstract

Recent technological advances have enabled distributed control systems to be implemented via networks. This allows feedback control loops to be closed over a shared communication channel. Network-induced delays are inevitable, however, when transmitting digital data between control devices. This paper analyzes the stability of such Networked Control Systems (NCS). We first review some previous work on this topic, offering some improvements. We analyze the influence of the sampling rate and network delay on system stability. We further study the stability of NCS using a hybrid system stability analysis technique.

## 1 Introduction

Real-time control networks are an emerging technology in industry which is receiving increasing attention. As available technologies (DeviceNet, Profibus, FireWire, etc.) become more mature and their prices decrease, they are being implemented in more real-time distributed control applications. One important feature of such NCS is that, instead of hardwiring the control devices with point-to-point connections, sensors, actuators, and controllers are all connected to the network as nodes. Several advantages of this implementation include: reduced system wiring, plug and play devices, increased system agility, and ease of system diagnosis and maintenance.

One issue inherent to NCS, however, is the network-induced delay that occurs while exchanging data among devices connected to the shared medium. This delay, either constant or time varying, can degrade the performance of control systems designed without considering it and even de-stabilize the system. Therefore, methods to analyze and design NCS are needed.

There are two main directions in approaching this problem. One way is to design the controller without regard to the delay and then design a communication protocol that minimizes delay or makes delay constant. For example, DeviceNet, which is based on CAN technology, uses a Carrier Sense Multiple Access with Bitwise

Arbitration (CSMA/BA) protocol to decrease the delay for higher priority messages; on the other hand, FireWire's isochronous transfer tries to make the delay constant by guaranteeing each node a time slot to transmit in every cycle. The other approach is to design control strategies that *a priori* compensate for the network-induced delay.

The situation is thus analogous to the design of sampled-data controllers: one can either design for the continuous-time case and make the sample period as small as possible—hoping for the best, or one can develop analysis and design procedures that explicitly take the sampled-data problem into account. In this paper, we begin to study NCS from this second point of view. Our investigation is organized as follows. We first review related work on the topic, offering some improvements. We then analyze how NCS implementation affects the stability of the system and how much delay the system can tolerate before it becomes unstable. We also apply a hybrid systems stability analysis technique to derive equations constraining the maximum allowable sampling period. Throughout,  $\|\cdot\|$  denotes the 2-norm for vectors and the induced 2-norm for matrices.

## 2 Review of Previous Work

Halevi and Ray [7] consider a continuous-time plant and discrete-time controller and analyze the Integrated Communication and Control System (ICCS) using a discrete-time approach. They study a clock-driven controller with *mis-synchronization* between plant and controller. The system is represented by an augmented state vector that consists of past values of the plant input and output, in addition to the current state vectors of the plant and controller. This results in a finite-dimensional, time-varying discrete-time model. They also take *message rejection* and *vacant sampling* into account.

Nilsson [6] also analyzes NCS in the discrete-time domain. He further tries to model the network delay as constant delays, independent random delays, and random delays governed by underlying Markov chains.

From there, he solves the LQG optimal control problem for the different delay models. He also points out the importance of time-stamping messages, which allows the history of the system to be known.

In Walsh *et al.* [9], the authors consider a continuous plant and a continuous controller. The control network, shared by other nodes, is only inserted between the sensor and the controller. They introduce the notion of Maximum Allowable Transfer Interval (or MATI), which they denote by  $\tau$ . Their goal is to find that  $\tau$  under which the desired performance (e.g., stability) of NCS is guaranteed to be preserved.

It is assumed that the non-networked, feedback system,

$$\dot{x}(t) = A_{11}x(t), \quad x(t) = [x_p(t) \ x_c(t)]^T,$$

(where  $x_p$  and  $x_c$  represent the plant and controller state) is globally exponentially stable, and thus, there exists a  $P$  such that

$$A_{11}^T P + P A_{11} = -I. \quad (1)$$

Next, it is assumed that the network's effects can be computed by the error,  $e(t)$ , between the plant output and controller input. So the networked-system's state vector is:  $z(t) = [x(t) \ e(t)]^T$ , and thus, the networked, closed-loop system is

$$\dot{z}(t) = Az(t),$$

where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}. \quad (2)$$

Walsh *et al.* study two scheduling methods: Try-Once-Discard (TOD) and token-ring type static scheduling under one-packet as well as multiple-packet transmission cases. Their result for the bound on  $\tau$  with multiple-packet transmission is given in the following theorem.

**Theorem 1 (Theorem 2 [9])** *Given NCS with  $p$  nodes of sensors operating under TOD or static scheduling, define  $\lambda_1 = \lambda_{\min}(P)$ ,  $\lambda_2 = \lambda_{\max}(P)$  (where  $P$  was defined above). If the MATI satisfies*

$$\tau < \min \left\{ \frac{\ln(2)}{p\|A\|}, \frac{1}{8\|A\|(\sqrt{\lambda_2/\lambda_1} + 1) \sum_{i=1}^p i}, \frac{1}{16\lambda_2\sqrt{\lambda_2/\lambda_1}\|A\|^2(\sqrt{\lambda_2/\lambda_1} + 1) \sum_{i=1}^p i} \right\},$$

*then the NCS is globally exponentially stable.*

The calculation of the bound for the  $\tau$  can be generalized and tightened by the following corollary.

**Corollary 1** *If the Lyapunov function  $V(x) = x^T P x$  of the non-networked, closed-loop system satisfies*

$$A_{11}^T P + P A_{11} = -Q, \quad (3)$$

*(more general than (1)), where  $P, Q$  are positive definite symmetric matrices, the bound on  $\tau$  becomes*

$$\tau < \min \left\{ \frac{\ln(2)}{p\|A\|}, \frac{1}{8\|A\|(\sqrt{\lambda_2/\lambda_1} + 1) \sum_{i=1}^p i}, \frac{\lambda_{\min}(Q)}{16\lambda_2\sqrt{\lambda_2/\lambda_1}\|A\|^2(\sqrt{\lambda_2/\lambda_1} + 1) \sum_{i=1}^p i} \right\}.$$

*Furthermore, the 3rd term is always the smallest, so*

$$\tau < \frac{\lambda_{\min}(Q)}{16\lambda_2\sqrt{\lambda_2/\lambda_1}\|A\|^2(\sqrt{\lambda_2/\lambda_1} + 1) \sum_{i=1}^p i} \quad (4)$$

*guarantees global exponential stability of NCS.*

**Proof:** See Appendix. ■

Corollary 1 shows that the MATI  $\tau$  depends on  $\|A\|$ ,  $q$ , and  $Q$ ;  $Q$  in turns decides  $P$  using (3).  $\|A\|$  and  $q$  are fixed for a particular system setup, thus  $Q$  is the only variable in picking  $\tau$ . One might use an analytic method to find  $Q$  that could maximize  $\tau$ . By maximization we mean that the largest  $\tau$  possible that could still preserve stability of NCS. However, the following example illustrates using random search in picking  $\tau$ .

**Example 1** Consider the state-space plant model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u \quad (5)$$

$$y = [1 \ 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

A state feedback controller is  $u = -Kx$ , where  $K = [3.75, 11.5]$  (closed-loop poles at  $-1/2$  and  $-3/4$ ).

Using Theorem 1, for  $p = 1$  (only one network node, which is the non-networked sampled-data system), we obtain  $\tau = 2.7 \times 10^{-4}$  s. By randomly selecting  $Q$  and solving for  $P$ , we can calculate  $\tau$  using the formula in Corollary 1. In 200 trials, the maximum  $\tau$  found was  $4.5 \times 10^{-4}$  s. However, the maximum stable constant sampling period for this feedback control system is 1.7s, which shows that Theorem 1 and Corollary 1 may be conservative.

Theorem 1 and Corollary 1 give sufficient conditions on the network sampling rate in order to guarantee the original non-networked system remains stable when the control loop is closed over the network. However, they might be too conservative to be of practical use. We next examine stability for some specific examples in order to develop some insight into the problem.

### 3 Our NCS Model

Our NCS model consists of a continuous plant

$$\dot{x}(t) = Ax(t) + Bu(t)$$

and a discrete controller

$$u(k) = -Kx(k), \quad k = 0, 1, 2, \dots$$

as shown in Figure 1. Here,  $x \in R^n$ ,  $u \in R^m$ , and  $A, B, K$  are of compatible dimensions. Figure 1 shows a model of NCS. There are two sources of delays from the network: the sensor-controller delay,  $\tau_{sc}$ , and controller-actuator delay,  $\tau_{ca}$ . The controller computational delay can be absorbed into either  $\tau_{sc}$  or  $\tau_{ca}$  without loss of generality.

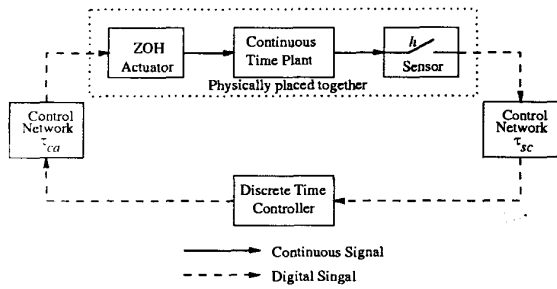


Figure 1: NCS Model

We consider two setups. In the first one, there are (a) clock-driven sensors which sample the plant outputs periodically at sampling instants; (b) an event-driven controller, which can be implemented by an external event interrupt mechanism and calculates the control signal as soon as the sensor data arrives; and (c) event-driven actuators, which mean the plant inputs are changed as soon as the data become available.

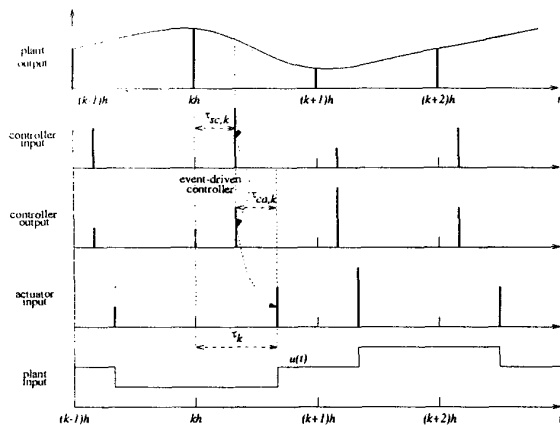


Figure 2: Timing of event-driven control

In the second setup, we consider a clock-driven controller, as in [7], that operates at the sampling instants.

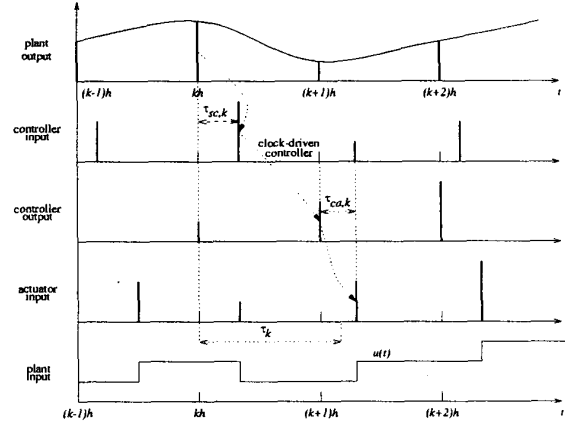


Figure 3: Timing of clock-driven control

The timing of signals of the two setups with  $\tau < h$  are shown in Figures 2 and 3. We can see that if the sensor-controller delay is a fraction of the sampling period, the clock-driven controller must wait until the next sampling instant to process the data. This potentially increases the delay, and motivates an event-driven controller.

Since the control law is fixed in both setups (time-invariant controller), the sensor-controller delay and controller-actuator delay can be lumped together as  $\tau = \tau_{sc} + \tau_{ca}$  for analysis purposes.

#### 3.1 Delay Less Than One Sampling Period

First consider the case where the delay of each sample,  $\tau_k$ , is less than one sampling period,  $h$ . Here, the subscript stands for the sampling instant. This constraint guarantees that at most two controls,  $u((k-1)h)$  and  $u(kh)$ , can be applied during the  $k$ th sampling period. The system equations can be written as:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad t \in [kh + \tau_k, kh + h + \tau_k), \\ y(t) &= Cx(t), \\ u(t^+) &= -Kx(t - \tau_k), \quad t \in \{kh + \tau_k, k = 0, 1, \dots\}. \end{aligned} \quad (6)$$

Sampling the system with period  $h$  we obtain [1]:

$$\begin{aligned} x(k+1) &= \Phi x(k) + \Gamma_0(\tau_k)u(k) + \Gamma_1(\tau_k)u(k-1), \\ y(k) &= Cx(k), \end{aligned}$$

where

$$\begin{aligned} \Phi &= e^{Ah}, \\ \Gamma_0(\tau_k) &= \int_0^{h-\tau_k} e^{As} ds B, \\ \Gamma_1(\tau_k) &= \int_{h-\tau_k}^h e^{As} ds B. \end{aligned}$$

Defining  $z(k) = [x(k), u(k-1)]^T$  as the augmented state vector, the augmented closed-loop system is

$$z(k+1) = \tilde{\Phi}(k)z(k); \quad (7)$$

where

$$\tilde{\Phi}(k) = \begin{bmatrix} \Phi - \Gamma_0(\tau_k)K & \Gamma_1(\tau_k) \\ -K & 0 \end{bmatrix}.$$

If the delay is constant, i.e.,  $\tau_k = \tau$  for  $k = 0, 1, \dots$ , the system is still time-invariant, which simplifies the system analysis. Thus, we can envision network protocols, such as token-ring scheduling, which provide constant delay. The next question is, how much delay the system can tolerate? (i.e., how fast should the network operate?)

Another observation is that the sensor-controller delay can be compensated by an estimator if the messages sent out by sensors are time-stamped (cf. [6]). Traditional one-step prediction estimation can compensate delays less than one sampling period, since the estimate of  $x(k)$  only depends on the value of  $y(k-1)$ .

### 3.2 Longer Delays

When the delays can be longer than one sampling period (say,  $0 < \tau_k < lh$ ,  $l > 1$ ), one may receive zero, one, or more than one (up to  $l$ ) control input(s) in a single sampling period. In the special case where  $(l-1)h < \tau_k < lh$  for all  $k$ , one control input is received every sample period for  $k > l$ . In this case, the analysis follows that in [1], resulting in

$$\tilde{\Phi}(k) = \begin{bmatrix} \Phi & \Gamma_1(\tau'_k) & \Gamma_0(\tau'_k) & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -K & 0 & 0 & \dots & 0 \end{bmatrix}, \quad (8)$$

where  $\tau'_k = \tau_k - (l-1)h$  and the augmented state vector is  $z(k) = [x(k), u(k-l), \dots, u(k-1)]^T$ .

In the more general case, there is tedious bookkeeping to perform as the matrix  $\tilde{\Phi}$  is time-varying since it depends on the exact schedule of the receipt of the control inputs.

## 4 Stability Regions for NCS

Normally, a faster sampling rate in a sampled-data system is desirable so the discrete-time control design can approximate more closely the continuous system. But in NCS, a faster sampling rate can increase the network load, which in turn results in longer delay of the signals. Thus, finding a sampling rate that can both tolerate the network-induced delay and achieve desired system performance is important in NCS design.

Plotting the stability region of NCS with respect to the sampling rate,  $h$ , and network delay,  $\tau$ , is helpful to see the relations between these two parameters. Note that we are considering constant delay here, which can be achieved by using the appropriate network protocol.

### 4.1 Integrator Case

Relations between  $h$  and  $\tau$  can be derived analytically for simple scalar systems.

**Example 2** Consider the *integrator* example:

$$\begin{aligned} \dot{x}(t) &= u(t-\tau), & (\tau < h), \\ u(kh) &= -Kx(kh), & (K > 0). \end{aligned} \quad (9)$$

Defining  $z(t) = [x(k), u(k-1)]^T$  as the augmented state vector, the augmented closed-loop system is

$$z(k+1) = \tilde{\Phi}z(k),$$

where

$$\tilde{\Phi} = \begin{bmatrix} 1 - hK + \tau K & \tau \\ -K & 0 \end{bmatrix}.$$

For this  $2 \times 2$  case, we can use the stability triangle [4] to explicitly calculate the relation between  $\tau$  and  $h$ .

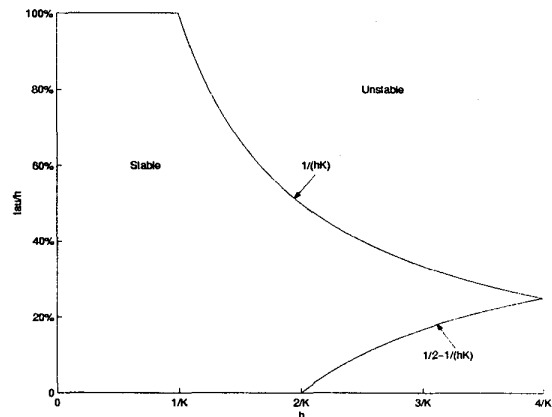
For a stable NCS, the delay  $\tau$  must satisfy:

$$\max \left\{ \frac{1}{2}h - \frac{1}{K}, 0 \right\} < \tau < \min \left\{ \frac{1}{K}, h \right\}, \quad (10)$$

or

$$\max \left\{ \frac{1}{2} - \frac{1}{Kh}, 0 \right\} < \frac{\tau}{h} < \min \left\{ \frac{1}{Kh}, 1 \right\}.$$

The stability region is determined analytically in Figure 4. We can see from the stability region that when the sampling period  $h$  is small, the system can tolerate a delay up to one full sampling period. As  $h$  becomes larger, the bound on  $\tau$  becomes smaller. An interesting observation is that for certain large values of  $h$ , the lower bound on  $\tau$  is not zero. Thus increasing the delay can occasionally be helpful.



**Figure 4:** Stability region of  $\dot{x} = u(t-\tau)$  ( $\tau < h$ ).

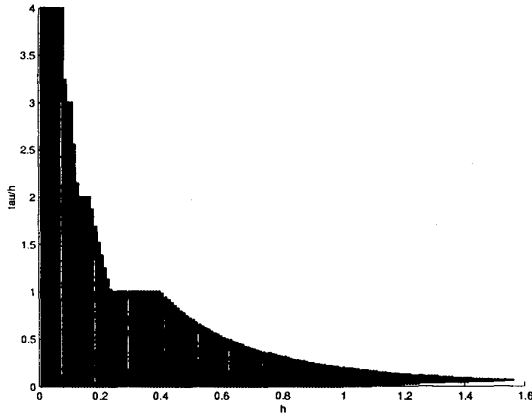
### 4.2 General Scalar System

It may be analytically infeasible to derive the exact stability region for general systems. However stability regions for such systems can still be determined by simulation. The stability region is plotted by increasing the delay  $\tau$  step by step, and testing the closed-loop system matrix, as formulated in Equations (7) and (8).

**Example 3** For a general scalar system

$$\begin{aligned} \dot{x}(t) &= ax(t) + u(t - \tau), \\ u(kh) &= -Kx(kh), \end{aligned}$$

the stability region can be determined by simulation. A special scalar case with  $a = 1$  and  $K = 2$  is shown in Figure 5. For this simulation, delay greater than one sampling period is also plotted, and we can see that when  $0 < \tau < h$ , it has the same shape as the integrator case. The shape of the stability region is also affected by the feedback controller (in this case, the feedback gain).



**Figure 5:** Simulation of the stability region of  $\dot{x}(t) = x(t) + u(t - \tau)$  with  $K = 2$  and  $0 < \tau < 4h$ .

### 5 Analyzing NCS Stability using a Hybrid Systems Technique

The stability of NCS can also be analyzed using hybrid systems stability analysis techniques. Hybrid systems contain continuous dynamics and discrete events [2]. The NCS model we are studying resembles a class of hybrid systems with fixed instants of impulse effect. The stability of such continuous-discrete systems was reviewed and extended in [3], where linearized hybrid systems of the following form are considered:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + f(x(t), u(t), t), \quad t \in I \setminus \Theta, \\ u(t^+) &= Cx(t) + Du(t) + \phi(x(t), u(t), t), \quad t \in \Theta, \end{aligned} \quad (11)$$

where  $x \in R^n, u \in R^m$ , and  $\Theta = \{t_k \mid t_k = kh, h > 0, k = 0, 1, \dots\}$ . Let  $z(t) = [x(t), u(t)]^T$ , then  $f(z, t) : \Omega_0 \times I \rightarrow R^n$  is continuous in  $z$  in the neighborhood  $\Omega_0 \subset R^{n+m}$  for any  $t$  in the interval  $I \subset R_+$ . Furthermore,  $f(0, t) = 0, t \in I, \phi(0, t) = 0, t \in \Theta$  and for  $z', z'' \in \Omega_0$ , the conditions

$$\begin{aligned} \|f(z', t) - f(z'', t)\| &\leq L_1 \|z' - z''\|^{1+\alpha}; L_1, \alpha > 0, t \in I, \\ \|\phi(z', t) - \phi(z'', t)\| &\leq L_2 \|z' - z''\|^{1+\alpha}; L_2, \alpha > 0, t \in \Theta, \end{aligned}$$

hold. The stability of this type of system comes down to evaluating the Schur-ness (i.e., whether all the eigenvalues of a matrix have magnitude less than one) of

$$H = \begin{bmatrix} e^{Ah} & \tilde{B} \\ Ce^{Ah} & C\tilde{B} + D \end{bmatrix},$$

where  $\tilde{B} = \int_0^h e^{A(h-s)} ds B \equiv E(h)B$ .

**Theorem 2 (Corollary 14 [3])** *If  $H$  is Schur, then the zeroth solution of (11) is asymptotically stable.*

We now apply this to NCS. Using the NCS model in (6) and referring to the timing of the signals of the event-driven controller shown in Figure 2, the following is suitable for our analysis:

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) - BK\hat{x}(t), \quad t \in [kh + \tau, (k+1)h + \tau), \\ \hat{x}(t^+) &= x(t - \tau), \quad t \in \{kh + \tau, k = 0, 1, \dots\}. \end{aligned} \quad (12)$$

Write  $x(t - \tau)$  in terms of  $x(t)$  and  $\hat{x}(t)$

$$x(t - \tau) = e^{-A\tau}x(t) + e^{-A\tau}E(\tau)BK\hat{x}(t),$$

and set

$$\begin{aligned} C &= e^{-A\tau}, \\ D &= e^{-A\tau}E(\tau)BK. \end{aligned}$$

Comparing this to (11), we obtain:

**Corollary 2** *The stability of NCS comes down to examining the Schur-ness of*

$$H = \begin{bmatrix} e^{Ah} & -E(h)BK \\ e^{A(h-\tau)} & -e^{-A\tau}(E(h) - E(\tau))BK \end{bmatrix}.$$

We can use this to re-calculate the integrator example of (9). By setting  $A = 0$  and  $B = 1$  in (12), we find

$$H = \begin{bmatrix} 1 & -hK \\ 1 & -(h - \tau)K \end{bmatrix}.$$

In order for  $H$  to be Schur,  $\tau$  must satisfy (10), which verifies the previous example.

### 6 Conclusion

A NCS model consisting of clock-driven sensors and event-driven controller and actuators is studied in this paper. The timing of such a model is also compared to the model with a clock-driven controller. Some previous stability results were sharpened. The relations between the sampling rate  $h$  and the network-induced delay  $\tau$  is analyzed using a stability region plot. Stability of NCS is also characterized using a hybrid systems stability analysis technique.

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## Appendix

**Proof:** The Lyapunov function  $V(x)$  must satisfy the following inequalities

$$\lambda_1 \|x\|^2 \leq V(x) \leq \lambda_2 \|x\|^2,$$

$$\dot{V}(x) = -X^T Q X \leq -\lambda_{\min}(Q) \|x\|^2.$$

Thus, following the proof of Theorem 2 in [9], Equation (8) of [9] should be re-written as:

$$\dot{V}(x(t)) \leq \|x(t)\|(-\lambda_{\min}(Q)\|x(t)\| + 2\lambda_2 \|A\| \gamma_1 \|z(t_0)\|).$$

Because of the choice of  $\tau$ ,  $\gamma_1 < \frac{\lambda_{\min}(Q)}{8\lambda_2 \sqrt{\lambda_2/\lambda_1} \|A\|} \leq \frac{\lambda_{\min}(Q)}{8\lambda_2 \|A\|}$ . Then, the same result as in [9] can be obtained: for all  $t > t_0 + p\tau$ ,  $V(x(t)) < \gamma_2 \|z(t_0)\|^2$ .

Setting  $\gamma_1 < \min\left\{\frac{1}{4}, \frac{\lambda_{\min}(Q)}{8\lambda_2 \sqrt{\lambda_2/\lambda_1} \|A\|}\right\}$ , we still have  $V(x(t)) < \gamma_2 \|z(t_0)\|^2$ ,  $\|e(t)\| < \gamma_1 \|z(t_0)\|$  for all  $t > t_0 + p\tau$ .

Viewing the NCS as perturbed by the bounded error signal  $e(t)$ , consider the system

$$\dot{x}_z(t) = A_{11}x_z(t) + A_{12}e(t)$$

starting at  $t = t_0 + p\tau$  with zero initial condition  $x_z(t_0 + p\tau) = 0$ . We can conclude that for all  $t > t_0 + p\tau$ ,

$$V_z(x_z(t)) < \frac{4\lambda_2^3 \|A\|^2 \gamma_1^2 \|z_0(t)\|^2}{\lambda_{\min}^2(Q)},$$

$$\|x_z(t)\| < \frac{2\lambda_2 \sqrt{\lambda_2/\lambda_1} \|A\| \gamma_1 \|z(t_0)\|}{\lambda_{\min}(Q)},$$

and, by the choice of  $\gamma_1$  above,  $\|x_z(t)\| < \frac{1}{4} \|z(t_0)\|$ . The rest of the proof for this part continues as in [9].

We now precede to prove the 3rd bound is always the smallest among the three terms above. We first prove

$$\frac{1}{8\|A\|(\sqrt{\lambda_2/\lambda_1} + 1) \sum_{i=1}^p i} < \frac{\ln(2)}{p\|A\|}.$$

We know that  $\lambda_2/\lambda_1 \geq 1$ ,  $\sum_{i=1}^p i = p(p+1)/2$  and  $p \geq 1$ , hence

$$\frac{1}{8\|A\|(\sqrt{\lambda_2/\lambda_1} + 1) \sum_{i=1}^p i} \leq \frac{1}{8\|A\|p(p+1)} < \frac{1}{8p\|A\|}.$$

Since  $\ln(2)$  is greater than 0.125,

$$\frac{\ln(2)}{p\|A\|} > \frac{1}{8p\|A\|}$$

and the result follows. We then can prove

$$\begin{aligned} & \frac{\lambda_{\min}(Q)}{16\lambda_2 \sqrt{\lambda_2/\lambda_1} \|A\|^2 (\sqrt{\lambda_2/\lambda_1} + 1) \sum_{i=1}^p i} \\ & \leq \frac{1}{8\|A\|(\sqrt{\lambda_2/\lambda_1} + 1) \sum_{i=1}^p i}. \end{aligned}$$

Canceling common terms, we must show

$$\frac{\lambda_{\min}(Q)}{2\lambda_2 \sqrt{\lambda_2/\lambda_1} \|A\|} \leq 1.$$

Since  $\sqrt{\lambda_2/\lambda_1} \geq 1$  by definition, it is enough to show

$$\frac{\lambda_{\min}(Q)}{2\lambda_2 \|A\|} \leq 1.$$

Taking norms on both sides of (3), we obtain

$$\begin{aligned} \|-Q\| &= \|A_{11}^T P + P A_{11}\| \\ &\leq \|A_{11}^T\| \|P\| + \|P\| \|A_{11}\| \\ &= 2\|P\| \|A_{11}\|, \end{aligned}$$

where the inequality follows from the triangle inequality plus the sub-multiplicative property of the matrix 2-norm. Now, since  $P$  is positive definite symmetric,  $\|P\| = \lambda_{\max}(P) = \lambda_2$ . Also,  $\|A\| \geq \|A_{11}\|$  follows easily from the definition of the induced norm, since the latter is a sub-matrix of the former (cf. (2)). Therefore,

$$\lambda_{\min}(Q) \leq \|-Q\| \leq 2\lambda_2 \|A\|$$

and

$$\frac{\lambda_{\min}(Q)}{2\lambda_2 \|A\|} \leq 1.$$