

Stability of Hybrid Systems: State of the Art

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Abstract

This paper collects work on the stability analysis of hybrid systems. The hybrid systems considered are those that combine continuous dynamics—represented by differential or difference equations—with finite dynamics—usually thought of as being a finite automaton. We review multiple Lyapunov functions as a tool for analyzing Lyapunov stability of general hybrid systems. Background results, the author's introductory work, and subsequent extensions are covered. Specializing to hybrid systems with linear dynamics in each constituent mode and linear jump operators, we review some key theorems of [4], and give corollaries encompassing several recently-derived “stability by first approximation” theorems in the literature. We also comment on the use of computational tests for stability of hybrid systems, and the general complexity thereof. The result is a tutorial on the state of the art in theory and computation of hybrid systems stability.

1 Introduction

We have in mind the following model as a prototypical example of a *switched system*:

$$\dot{x}(t) = f_q(x(t)), \quad q \in \{1, \dots, N\}, \quad (1)$$

where $x(t) \in R^n$. We add the following assumptions. (1) Each f_q is globally Lipschitz continuous. (2) The q 's are picked in such a way that there are finite switches in finite time.

Such systems are of “variable structure” or “multi-modal”; they are a simple model of (the continuous portion) of hybrid systems [1, 9]. We explain this below. The particular q at any given time may be chosen by some “higher process,” such as a controller, computer, or human operator, in which case we say that the system is *controlled*. It may also be a function of time or state or both, in which case we say that the system is *autonomous*. In the latter case, we may really just arrive at a single (albeit complicated) nonlinear, time-varying equation. However, one might gain some

leverage in the analysis of such systems by considering them to be amalgams of simpler systems.

A particular case of of interest for Equation (1) is the case of *switched linear systems*, where each of the f_q is a linear system:

$$\dot{x}(t) = A_q(x(t)), \quad q \in \{1, \dots, N\}, \quad (2)$$

where $x(t) \in R^n$.

In addition to the switching phenomenon discussed above, so-called *systems with impulse effect* often add the possibility of the state's jumping (also known as “resets”) when certain boundaries are crossed. In general, these boundaries are subsets of the space, M , but they may be given explicit representation in terms of the zeros of one or more functions.

$$\begin{aligned} \dot{z}(t) &= f(z(t)), & (z, t) \notin M_t, \\ z(t^+) &= J(z(t)), & (z, t) \in M_t \end{aligned}$$

The interpretation of the above is that the dynamics evolves according to the differential equation while (z, t) is in the complement of $M_t \subset Z \times I$, but that the state is immediately reset according to the map J upon the (z, t) 's hitting the set M_t . See [3, 9] for more details and conditions on when the dynamics is well-defined. There are three main cases of interest:

- Fixed Instants of Impulse Effect. The sets M_t are hyperplanes at fixed instants of $t = \tau_1, \tau_2, \dots$
- Mobile Instants of Impulse Effect. The sets M_t are a sequence of hypersurfaces $\sigma_k = \tau_k(x)$.
- Autonomous Impulse Effect. The sets M_t are constraints on the state space, i.e., they are of the form $M \times I, M \subset Z$.

To generalize both of the above to the case of *hybrid systems* combining continuous variables, $x \in R^n$, and a discrete variable, $q \in Z$, may take $z = (x, q)$ in Equation (3) and allow no differential update of the discrete variables q .

The paper is organized as follows. In the next section, we give some background stability results, encompassing switched linear systems and the use of single Lyapunov functions. Next, we examine a method which

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uses a number of different Lyapunov functions in order to prove stability. In Section 4, we discuss extensions of this “multiple Lyapunov method” and results on stability by first approximation of systems with fixed instants of impulse effect. In Section 5, we end by touching upon computational aspects, including tests for stability and complexity results.

2 Background Results

We assume the reader is familiar with basic Lyapunov theory [19]. Below, $S(r)$, $B(r)$, and $\bar{B}(r)$ represent the sphere, ball, and closed ball of Euclidean radius r about the origin in R^n , respectively.

Models like Equation (2) have been studied for stability [12, 23, 27, 13]. The former reference mainly considered switched systems where the sequence of matrices—and the times they are each applied—are periodic. They study stability, reachability, and observability of these periodic hybrid systems, and subsequently relax some results to the aperiodic case. As for aperiodic stability under some constraints, they give a sufficient condition based on estimates arising from the logarithmic norm (see [26] for definition) of the constituent matrices. In particular,

Theorem 1 (Theorem 1, [12]) *If matrix A_i is active for a fraction of time $p_i = \delta t_i/T$ in each time period of length T , then the hybrid system (2) is uniformly asymptotically stable if*

$$\sum_i \mu(A_i)p_i < 0,$$

where μ is a logarithmic matrix norm.

The theorem relies on the fact that over periods of time of length T , the system’s fundamental matrix of solutions is a contraction. Another way to show stability is to use Lyapunov functions, to which we now turn.

In the papers by [23, 27, 13], the question of stabilization of systems as in Equation (2) is studied. We follow [13] in summarizing their results.

Definition 2 *The system (2) is quadratically stabilizable via state-feedback if and only if there exists a positive definite function $V(x) = x^T P x$, $\epsilon > 0$, and a switching rule $q(x, t)$ such that*

$$\frac{d}{dt}V(x) < -\epsilon x^T x$$

for all trajectories of the system (2).

Theorem 3 ([27]) *The system (2) is quadratically stabilizable if there exists a convex combination of the A_i that is asymptotically stable.*

Theorem 4 (Theorem 2.2, [13]) *Assume $N = 2$. The system (2) is quadratically stabilizable if and only if there exists a convex combination of A_1 and A_2 that is asymptotically stable.*

In [3] and [22], more general hybrid systems are considered. The constituent systems considered need not be linear and impulsive jumps may occur. Most of the results for the former reference are in the case of fixed and mobile instants of impulse effects, but there is a section on stability of autonomous systems with impulse effect. Pavlidis considers systems in which there are “pulse firings” which impulsively reset the state. We merely wish to set the stage for multiple Lyapunov functions, discussed next. Hence, instead of giving full theoretical conditions (which are lengthy), we suffice to say that the *character* of these stability results is given by the following “theorem”:

Theorem 5 *If there exists a Lyapunov function that is non-increasing along continuous actions and non-increasing over each impulse, then the system is stable.*

The above may be internalized with the help of the cartoon in Figure 1. For details, see [10] and the original papers. The same idea, in the case of linear systems was used in [23]. We touch on these in the next section.



Figure 1: Lyapunov Stability for systems with Impulse Effect. Lyapunov function values versus time.

3 Multiple Lyapunov Functions

In this section, we discuss Lyapunov stability of switched systems via “multiple Lyapunov functions.” The idea here is that even if we have Lyapunov functions for each system f_q individually, we need to impose restrictions on switching to guarantee stability. Indeed, it is easy to construct examples of two globally exponentially stable systems and a switching scheme that sends all trajectories to infinity:

Example 6 *Consider $f_1(x) = Ax$ and $f_2(x) = Bx$ where*

$$A = \begin{bmatrix} -0.1 & 1 \\ -10 & -0.1 \end{bmatrix}, \quad B = \begin{bmatrix} -0.1 & 10 \\ -1 & -0.1 \end{bmatrix}.$$

Then $\dot{x} = f_q(x)$, is globally exponentially stable for $q = 1, 2$. But the switched system using f_1 in the second and fourth quadrants and f_2 in the first and third quadrants is unstable.

Below, we impose restrictions on switching sufficient to guarantee stability. We will make rigorous the following “theorem”:

Theorem 7 (Multiple Lyapunov Method)

Given N dynamical systems, $\Sigma_1, \dots, \Sigma_N$, each with equilibrium point at the origin, and N candidate Lyapunov functions, V_1, \dots, V_N .

If V_i decreases when Σ_i is active and

$$V_i \left(\begin{array}{c} \text{time when } \Sigma_i \\ \text{switched in} \end{array} \right) \leq V_i \left(\begin{array}{c} \text{last time } \Sigma_i \\ \text{switched in} \end{array} \right)$$

then the hybrid system is Lyapunov stable.

See Figure 2.

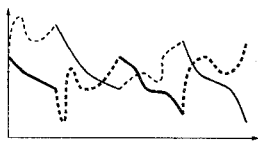


Figure 2: Multiple Lyapunov Stability. Lyapunov function values versus time.

For simplicity, we only consider the case where there is switching, but no impulsive jumps of the state, and where all equilibrium points are the the same. Relaxing the latter is possible and may lead to interesting applications [10]. Thus, we will be dealing with systems that switch among vector fields over time or regions of state-space. One can associate with such a system the following (*anchored*) *switching sequence*, indexed by an initial state, x_0 :

$$S = x_0; (q_0, t_0), (q_1, t_1), \dots, (q_N, t_N), \dots \quad (3)$$

The sequence may or may not be infinite. In the finite case, we may take $t_{N+1} = \infty$, with all further definitions and results holding. However, we present in the sequel only in the infinite case to ease notation. The switching sequence, along with Equation (1), completely describes the trajectory of the system according to the following rule: (q_k, t_k) means that the system evolves according to $\dot{x}(t) = f_{q_k}(x(t), t)$ for $t_k \leq t < t_{k+1}$. We denote this trajectory by $x_S(\cdot)$. Throughout, we assume that the switching sequence is *minimal* in the sense that $q_j \neq q_{j+1}$, $j \in \mathbb{Z}^+$.

We can take projections of this sequence onto its first and second coordinates, yielding the sequence of indices, $\pi_1(S) = x_0; q_0, q_1, \dots, q_N, \dots$, and the sequence

of switching times, $\pi_2(S) = x_0; t_0, t_1, \dots, t_N, \dots$, respectively. Suppose S is a switching sequence as in Equation (3). We will denote by $S|q$ the sequence of switching times whose corresponding index is q . The *interval completion* $\mathcal{I}(T)$ of a strictly increasing sequence of times $T = t_0, t_1, \dots, t_N, \dots$, is the set

$$\bigcup_{j \in \mathbb{Z}^+} (t_{2j}, t_{2j+1}).$$

Finally, let $\mathcal{E}(T)$ denote the *even* sequence of T : t_0, t_2, t_4, \dots .

Below, we say that V is a *candidate Lyapunov function* if V is a continuous, positive definite function (about the origin, 0) with continuous partial derivatives. Note this assumes $V(0) = 0$. We also use

Definition 8 (Lyapunov-like) Given a strictly increasing sequence of times T in \mathbb{R} , we say that V is a Lyapunov-like function for function f and trajectory $x(\cdot)$ over T if

- $\dot{V}(x(t)) \leq 0$ for all $t \in \mathcal{I}(t)$,
- V is monotonically nonincreasing on $\mathcal{E}(T)$.

Theorem 9 ([7]) Suppose we have candidate Lyapunov functions V_q , $q = 1, \dots, N$, and vector fields $\dot{x} = f_q(x)$ with $f_q(0) = 0$, for all q . Let \mathcal{S} be the set of all switching sequences associated with the system.

If for each $S \in \mathcal{S}$ we have that for all q , V_q is Lyapunov-like for f_q and $x_S(\cdot)$ over $S|q$, then the system is stable in the sense of Lyapunov.

Some remarks are in order:

- The case $N = 1$ is the usual theorem for Lyapunov stability [19].
- The theorem also holds if the f_q are time-varying.
- It is easy to see that the theorem does not hold if $N = \infty$, and we leave it to the reader to construct examples.

It is possible to use different conditions on the V_q to ensure stability. For instance, consider the following

Definition 10 If there are candidate Lyapunov functions V_q corresponding to f_q for all q , we say they satisfy the sequence nonincreasing condition for a trajectory $x(\cdot)$ if

$$V_{q_{j+1}}(x(t_{j+1})) < V_{q_j}(x(t_j)).$$

This is a stronger notion than the Lyapunov-like condition used above.

The sequence nonincreasing condition is used in the stability (version of the asymptotic stability) theorem of [23]. Thus that theorem is a special case of the continuous-time version of Theorem 9 above. Moreover, the proof of asymptotic stability in [23] is flawed since it only proves state convergence and not state convergence plus stability, as required. It can be fixed using our theorem.

Now, consider the case where the index set is an arbitrary compact set:

$$\dot{x} = f(x, \lambda), \quad \lambda \in K, \text{ compact.} \quad (4)$$

Here, $x \in R^n$ and f is globally Lipschitz in x , continuous in λ . For brevity, we only consider the continuous-time case. Again, we assume finite switches in finite time.

As above, we may define a switching sequence $S = x_0; (\lambda_0, t_0), (\lambda_1, t_1), \dots, (\lambda_N, t_N), \dots$ with its associated projection sequences.

Theorem 11 ([7]) *Suppose we have candidate Lyapunov functions $V_\lambda \equiv V(\cdot, \lambda)$ and vector fields as in Equation (4) with $f(0, \lambda) = 0$, for each $\lambda \in K$. Also, $V : R^n \times K \rightarrow R^+$ is continuous. Let S be the set of all switching sequences associated with the system.*

If for each $S \in S$ we have that for all q , V_λ is Lyapunov-like function for f_λ and $x_S(\cdot)$ over $S|\lambda$, and the V_λ satisfy the sequence nonincreasing condition for $x_S(\cdot)$, then the system is stable in the sense of Lyapunov.

This theorem is a different generalization of the aforementioned theorem in [23].

4 Extensions

The multiple Lyapunov method of the previous section can be used to prove the correctness of hybrid stabilization schemes discussed in the literature [2, 17]. In particular, the stabilization strategies proposed in [17], e.g., choosing at each time the minimum of several Lyapunov functions, clearly satisfies our switching condition. Theoretical extensions follow: (1) In proving stability, it is sometimes necessary to use more Lyapunov functions than constituent systems [24]. This extension is covered by the theorems above, however, by simply introducing new discrete sub-states with the same continuous dynamics but different Lyapunov functions. (2) In [28], the character of the theorems is to effectively relax the first part of our Lyapunov-like definition by

allowing the Lyapunov functions to increase in each active region if the gain from “switch in” to “switch out” is given by a positive definite function. This is certainly allowed as the definition of Lyapunov stability allows one to pick a strictly smaller neighborhood of initial conditions in staying within a given one. More provocatively, then, one may formulate relaxations allowing increases over energy V_q at switching times with a similar constraint from initial value to final limit. For other extensions and examples, see [10].

We now go to the case of hybrid systems with fixed instants of impulse effect. We review some theorems of [4] and give some corollaries encompassing results of [16] and [20]. In [4], Barabanov and Starozhilov consider the stability of class of “continuous-discrete systems.” To be specific, these are systems with fixed instants of impulse effect, as described in the Introduction. Focusing on systems with linear dynamics, linear impulses, and periodic instants of impulse effect, they prove several stability results, including a necessary and sufficient Lyapunov (second method) theorem. With this in hand, they turn to consider stability and instability by first approximation. In particular, they consider the system of equations

$$\begin{aligned} \dot{z} &= A_1 z + f(z, t), & t \in I/\Theta, \\ z(t^+) &= A_2 z(t) + \phi(z(t), t), & t \in \Theta, \end{aligned} \quad (5)$$

where $z \in R^n$, A_1 and A_2 are constant $n \times n$ matrices, $\Theta = \{t_k \mid t_k = kh, h > 0, k = 1, 2, \dots\}$, and the function $f(z, t) : \Omega_0 \times I \rightarrow R^n$ is continuous in z in the region Ω_0 for any $t \in \Theta$. Furthermore, $f(0, t) = 0$, $t \in I$, $\phi(0, t) = 0$, $t \in \Theta$ and for $z', z'' \in \Omega_0$, the conditions

$$\|f(z', t) - f(z'', t)\| \leq L_1 \|z' - z''\|^{1+\alpha}; \quad L_1, \alpha > 0, t \in I,$$

$$\|\phi(z', t) - \phi(z'', t)\| \leq L_2 \|z' - z''\|^{1+\alpha}; \quad L_2, \alpha > 0, t \in \Theta,$$

hold. Finally, they define

$$P = A_2 \exp\{A_1 h\}.$$

Using inequalities and lemmas derived in their paper, they conclude the following

Theorem 12 (Theorem 5, [4]) *If P is Schur (i.e., the eigenvalues of the matrix P are such that $|\lambda_i(P)| < 1$, $i = 1, \dots, n$), then the zeroth solution of Equation (5) is asymptotically stable.*

Theorem 13 (Theorem 6, [4]) *If P is strictly not Schur (i.e., $|\lambda_i(P)| > 1$ for even one eigenvalue of the matrix P), then the zeroth solution of Equation (5) is unstable.*

Now suppose we want to apply the above theorems in the case of hybrid systems of the following form,

$$\begin{aligned} \dot{x} &= Ax + Bq + f(x, q), & t \in [kT, (k+1)T) \\ q(t^+) &= Cx + Dq + \phi(x(t), q), & t \in kT, k = 0, 1, 2, \dots \end{aligned} \quad (6)$$

where f, ϕ have properties as defined above and all unspecified variables are evaluated at time t . Comparing to the foregoing, we may simply compute for this special case:

$$\begin{aligned} z &= \begin{bmatrix} x \\ q \end{bmatrix}, \\ A_1 &= \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} I & 0 \\ C & D \end{bmatrix}. \end{aligned}$$

In this case, the matrix P of [4] computes to

$$P = \begin{bmatrix} e^{AT} & \tilde{B} \\ Ce^{AT} & C\tilde{B} + D \end{bmatrix} \equiv H,$$

where $\tilde{B} = \int_0^T e^{A(T-\tau)} d\tau B \equiv E(T)B$.

Corollary 14 *If H is Schur, then the zeroth solution of Equation (6) is asymptotically stable.*

Corollary 15 *If H is strictly not Schur, then the zeroth solution of Equation (6) is unstable.*

4.1 Applications

Let us now consider the following *nonlinear sampled-data system*, as defined in [16]:

$$\begin{aligned} \dot{x} &= f(x(t)) + Bu(k), & t \in [k, k+1), \\ u(k+1) &= Cu(k) + Dx(k), & k = 0, 1, 2, \dots \end{aligned} \quad (7)$$

where $f(0) = 0$. First, let A denote the Jacobian of f evaluated at $x = 0$, $T = 1$, and $q(k) = u(k)$. Thus, converting to the form of Equation (6) above, the nonlinear sampled-data system becomes

$$\begin{aligned} \dot{x} &= Ax + Bq + \tilde{f}(x), & t \in [k, k+1), \\ q(k) &= Cq(k-1) + Dx(k-1), & k = 0, 1, 2, \dots \end{aligned} \quad (8)$$

But, we may compute from the first of these equations that

$$x(k-1) = e^{-A}x(k) - e^{-A}E(1)Bq(k-1).$$

Hence, the shift of time and matrix notation means we must set

$$\begin{aligned} C_{(6)} &= D_{(7)}e^{-A}, \\ D_{(6)} &= C_{(7)} - D_{(7)}e^{-A}E(1)B. \end{aligned}$$

By our corollaries above, stability of Equation (7) comes down to examining Schur-ness of

$$H = \begin{bmatrix} e^A & E(1)B \\ D_{(7)} & C_{(7)} \end{bmatrix}.$$

This is exactly the matrix in Equation (6) of [16]. Hence, one sees that Corollaries 14 and 15 encompass Theorems 1 and 2 of [16] as special cases.

As another application of the corollaries above, we consider the following *general hybrid system*, as defined in [20]:

$$\dot{x} = g(x(t), x(kT)), \quad t \in [kT, (k+1)T),$$

where $g : R^{2n} \rightarrow R$ is continuously differentiable with $g(0, 0) = 0$. Hence we may represent g as

$$g(x, z) = Ax + Bz + \tilde{g}(x, z),$$

where A and B are the Jacobian matrices at $(0, 0)$ with respect to x and z , respectively. Thus, setting $q(kT) = x(kT)$, we have

$$\begin{aligned} C_{(6)} &= I, \\ D_{(6)} &= 0. \end{aligned}$$

So, H of our corollaries reduces in this case to

$$\begin{bmatrix} e^{AT} & \tilde{B} \\ e^{AT} & \tilde{B} \end{bmatrix},$$

which is Schur if $e^{AT} + \tilde{B}$ is Schur. Thus, Theorem 1 of [20] is also a special case of the Corollary 14.

5 Computational Aspects

5.1 Stability Tests

Computational tests for stability of switched linear systems appear in [18, 24, 13]. Each use the case of linear constituent systems and *autonomous switching* (i.e., identity reset maps, yielding no jumps in states), as in Equation (2). Each develop computational tests for stability based on linear matrix inequalities. [13] builds on the results of [23, 27], while [18, 24] build on [7]. These latter two actually treat affine constituent systems.

5.2 Complexity

Even simple hybrid control problems can exhibit complicated, even chaotic behavior [11]. Furthermore, questions of the stability of hybrid systems in as little as three dimensions (i.e., with continuous state space R^3) have been shown to be computationally undecidable [8]. However, those results considered general continuous (viz. nonlinear) vector fields. Recently, complexity and undecidability results have been achieved in the case

where the constituent systems are each linear [25, 5, 6], as in Equation (2). These results continue to be undecidable even if there are only two linear systems, with switching occurring on a hyperplane through the origin. The results in these papers have their roots in a paper by Paterson [21]. There, Paterson considers the problem of *mortality* of a set S of 3×3 matrices, i.e., whether there is a finite product of members of S that equals the zero matrix. He reduces this problem to Post's Correspondence Problem, perhaps the most famous undecidable problem after the Halting Problem [15]. NP-completeness may then also be derived by considering bounded versions of Post's Correspondence Problem [14].

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