Proving Languages Equal or Not

Spring 2005, Prof. Branicky

To prove languages are not equal, provide a counter-example, that is a string that is one but not the other. To prove that languages are equal, you must show each is a subset of the other. That is, pick a string in the first, and show it is in the second, and vice versa.

Example 1

Question: Is $(a+b)^* = a^* + b^*$? **Answer:** No, because $ab \in (a+b)^*$ but $ab \notin a^* + b^*$.

Example 2

Question: Is $\overline{L^*} = (\overline{L})^*$ for all languages L? **Answer:** Let's see.

First, the statement is true if $L = \emptyset$, because $\emptyset^* = \{\epsilon\}$, $\overline{\emptyset} = \emptyset$, and $\overline{\{\epsilon\}} = \{\epsilon\}$. If L is non-empty, the proof of equality would have two parts.

Part I: Show $\overline{L^*} \subseteq (\overline{L})^*$. Pick a string $x \in \overline{L^*}$. We wish to show that it is in $(\overline{L})^*$. Since $x \in \overline{L^*}$, it is the prefix of a string $w \in L^*$. There are two cases to consider:

1. w, and hence x, are both the empty string. In this case, $x \in (\overline{L})^*$, since the star of any language contains the empty language.
2. $w = w_1 \cdots w_n$, $n \geq 1$, and $w_i \in L$ for all i. In this case, it is clear that prefix x is of the form
$$w_1 \cdots w_j v$$
where $0 \leq j < n$ and v is a prefix of w_{j+1} (in the case $j = 0$, $w_1 w_0 = \epsilon$ and v is a prefix of w_1). But then we see that x is a concatenation of $j+1$ strings, each of which are in $\overline{L}$, so that $x \in (\overline{L})^*$.

Thus, Part I is done and $\overline{L^*} \subseteq (\overline{L})^*$.

Part II: Show $\overline{L^*} \supseteq (\overline{L})^*$. Here, we begin with a string $x \in (\overline{L})^*$. There are two cases to consider:

1. $x = \epsilon$. In which case, $x \in \overline{L^*}$.
2. $x = x_1 \cdots x_n$ where $n \geq 1$ and $x_i \in \overline{L}$ Is this always a member of $\overline{L^*}$? Doesn't seem like it if each x_i is a proper prefix. See below.

Thinking about the above, I came up with the following counter-example. Let $L = \{ab\}$, that is, a language containing only one string. Then, $a \in \overline{L}$ and $aa \in (\overline{L})^*$. Now,
$$L^* = \{\epsilon, ab, abab, ababab, \ldots\}$$
One can see that aa is not a prefix of any of these, because any prefix of length two has b as its second letter. Thus, we have shown that in general $\overline{L^*} \not\supseteq (\overline{L})^*$ and $\overline{L^*} \neq (\overline{L})^*$. **Note**: The counter-example alone is proof of inequality; the "scratch work" above is included to show how each part of the equality is proven.