

EECS 343 E-Handout: Regular or Not?

Prof. Branicky

To Prove a Language is Regular:

- Synthesize a DFA,
- Synthesize an NFA,
- Synthesize an ϵ -NFA,
- Synthesize a regular expression,
- Use closure properties, **or**
- Use the Homomorphism Theorems

To Prove a Language is Not Regular

- Find a counterexample to the Pumping Lemma,
- Find a counterexample to the closure properties, **or**
- Find a counterexample to the Homomorphism Theorems

Examples

(a) $\{w1^n \mid w \in \{0, 1\}^*, n \geq 0\}$

This is regular. A regex is: $(0+1)^*1^*$

(b) The *parameterized family* of languages $L_n = \{w1^n \mid w \in \{0, 1\}^*\}$, for $n \geq 0$.

Each of these is regular: regex for L_n is $(0+1)^*1^n$. In particular, $L_5 = (0+1)^*11111$.

(c) $\{w1^n \mid w \in \{0, 1\}^*, |w| = n, n \geq 0\}$

Preliminaries. What is this language? Some strings that are members:

$$0^91^9, \quad 1^{18}, \quad 01^{17}, \quad 01011111, \quad 0011, \quad 0111, \quad 1011, \quad 1111$$

In English, these are even-length strings whose “last half” is all 1’s.

A False Start. This language is not regular. Here is my counterexample to the pumping lemma. For arbitrary p , I’ll pick

$$s = 1^p1^p$$

and split it as follows:

$$x|y|z = 1^p|1^p|\epsilon$$

Now, when the demon picks v , I’ll pump it down (choose $i = 0$) and get a string of the form

$$1^p1^r$$

where $r < p$, which isn’t the pattern. Contradiction. The language is not regular.

This is **wrong!** For v of odd length, the string is not in the language; however, for v of even length, it can be “reinterpreted” as

$$1^p 1^r = 1^p 1^{p-|v|} = 1^{p-|v|/2} 1^{p-|v|/2},$$

which *is* in the language.

A Roadblock. This language is not regular. Here is my counterexample to the pumping lemma. For arbitrary p , I’ll pick

$$s = 0^{p-1} 11^p$$

and split it as follows:

$$x|y|z = 0^{p-1} 1|1^p|\epsilon$$

Now, you must find a counterexample that works for any $v \neq \epsilon$. Can you find one if $|v|$ is even? If $|v| = 2$?¹ Well, I still have a lot of choice; I get to choose i . Try $i = 0$. Then,

$$s_0 \triangleq x(uv^0w)z = 0^{p-1} 11^{p-2} = 0^{p-1} 1^{p-1}$$

Ugh, it’s in the language. OK, I’ll choose $i = 2$:

$$s_2 \triangleq x(uv^2w)z = 0^{p-1} 11^{p+4} = 0^{p-1} 1111^{p+2}$$

Can I ever win? If I choose $i = n$, then

$$s_n \triangleq x(uv^nw)z = 0^{p-1} 11^{p+2n} = 0^{p-1} 11^n 1^{p+n}$$

Unfortunately, this is also in the language since it has length $2(p+n)$ and its last $p+n$ symbols are 1’s.

So, that particular s does not work as a counterexample. And that’s all we know.

Correct Proof. This language is not regular. Here is my counterexample to the pumping lemma. For arbitrary p , I’ll pick

$$s = 0^p 1^p$$

and split it as follows:

$$x|y|z = 0^p |1^p| \epsilon$$

Now, for arbitrary split of $y = uvw$, with $|v| \geq 1$, I will choose $i = 0$. Then,

$$x(uv^0w)z = 0^p 1^{p-|v|}$$

This is not of the correct form (it may have even length, but it doesn’t have enough 1’s). This is a counterexample.²

Bottom line: The language is not regular.

¹In the game formulation, do you still have a winning strategy if the demon chooses $|v| = 2$? By the way, he was sure to have picked $p \geq 2$ in his first step.

²In the game formulation, it is a (parameterized) description for a winning strategy (that works no matter what p or v is chosen by the demon).

(d) $L = \{0^i 1^j \mid i \neq j\}$

Assume L is regular. Now, $M = \{0^* 1^*\}$ is regular. Thus, by closure properties, $M - L$ is regular. But, $M - L = \{0^i 1^i \mid i \geq 0\} = B$, a language we defined and proved was not regular in lecture. Contradiction. So, L is not regular.

(e) $L = \{w \in \{0, 1\}^* \mid |w| \text{ is a perfect cube}\}$.

Assume L is regular. Define homomorphism h by

$$h(\epsilon) = \epsilon, \quad h(0) = 1, \quad h(1) = 1$$

Then $h(L)$ is regular by the Homomorphism Theorems. But, $h(L) = \{1^{n^3} \mid n \geq 0\} = T$, a language we defined and proved was not regular in lecture. Contradiction. So, L is not regular.

(f) $L = \{w \in \{0, 1\}^* \mid \#0(w) > \#1(w)\}$.

Assume L is regular. Then by closure properties, so is

$$\sim L = \{w \in \{0, 1\}^* \mid \#0(w) \leq \#1(w)\}$$

and

$$A \triangleq \sim L \cap \{0^* 1^*\} = \{0^m 1^n \mid n \geq m \geq 0\}$$

Define h by

$$h(\epsilon) = \epsilon, \quad h(0) = 1, \quad h(1) = 0$$

Then by the Homomorphism Theorems and closure properties,

$$A' \triangleq \mathbf{rev}(h(A)) = \{0^n 1^m \mid n \geq m \geq 0\}$$

is also regular. But, then so should

$$A \cap A' = B,$$

again the language we defined and proved was not regular in lecture. Contradiction. So, L is not regular.

(g) Let $\Sigma = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \dots, \mathbf{z}\}$. Define

$$M = \{x \in \Sigma^* \mid \# \text{ of vowels} > \# \text{ of consonants}\}$$

Some strings in M include

a, ooo, air, bee, beekeeper, delicious, radio

Assume M is regular. Define $h : \Sigma^* \rightarrow \{0, 1\}^*$ defined by

$$h(\epsilon) = \epsilon, \\ h(\sigma) = \begin{cases} 0, & \sigma \in \{\mathbf{a}, \mathbf{e}, \mathbf{i}, \mathbf{o}, \mathbf{u}\}, \\ 1, & \text{otherwise.} \end{cases}$$

Then, $h(L)$ is regular. But $h(M) = L$, the language defined and proved not regular in the previous example. Contradiction. So, M is not regular.