EEAP 246
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z - TRANSFORM

LAST TIME

- MOTIVATION
  - A TOOL FOR ANALYZING D-T SYSTEMS
  - APPLICATIONS IN DIG. CONTROL, DIG. SIGNAL PROC.
- DEFINITION / MEANING
- PROPERTIES

THIS TIME

- REVIEW
- THE INVERSE z-TRANSFORM
- A FAST WAY TO SOLVE DIFFERENCE EQUATIONS

NEXT TIME(S)

- USE TRANSFER FCN. IDEAS TO SOLVE D-T SYSTEM
- DIG. CONTROL & DIGITAL FILTERING

(UNILATERAL) z - TRANSFORM

TAKES A SEMI-INFINITE SEQUENCE, $x[n]$, $n = 0, 1, 2, \ldots$
TO A SERIES OF A VARIABLE z, $X(z)$

$$X(z) = \sum_{n=0}^{\infty} \frac{x[n]}{z^n}$$

① FCN OF z: $\quad X(1) = \sum_{n=0}^{\infty} x[n]$, $\quad X(2) = \sum_{n=0}^{\infty} \frac{x[n]}{2^n}, \ldots$

② "z" IS A PLACEHOLDER THAT SIGNIFIES A NUMBER'S PLACE IN A SEQUENCE. EQUATE TWO SEQUENCES BY EQUATING COEFF.'S OF POWERS OF z.

E.G. $\quad \dfrac{z}{z-1} = 1 + \dfrac{1}{z} + \dfrac{1}{z^2} + \dfrac{1}{z^3} + \cdots$

$$\begin{array}{r|l}
 & 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots \\ \hline
z-1 & z \\
 & \underline{z-1} \\
 & \quad 1 \\
 & \quad \underline{1 - \frac{1}{z}} \\
 & \qquad \frac{1}{z} \\
 & \qquad \underline{\frac{1}{z} - \frac{1}{z^2}} \\
 & \qquad\quad \frac{1}{z^2} \\
 & \qquad\quad \underline{\frac{1}{z^2} - \frac{1}{z^3}} \\
 & \qquad\qquad \ddots
\end{array}$$

SIMPLE PAIRS

$$a^n u[n] \longleftrightarrow \frac{1}{1 - a z^{-1}} = \frac{z}{z-a} \qquad \left(= \frac{a^{-1} z}{a^{-1} z - 1}\right)$$

$$u[n] \longleftrightarrow \frac{1}{1 - z^{-1}} = \frac{z}{z-1}$$

$$\delta[n] \longleftrightarrow 1$$

$$\delta[n-c] \longleftrightarrow z^{-c} = \frac{1}{z^c}, \qquad c \geq 0$$

Z-TRANSFORM PROPERTIES

LINEARITY: $a\,x[n] + b\,y[n] \longleftrightarrow a\,X(z) + b\,Y(z)$

MULT. BY EXPONENTIAL: $a^n x[n] \longleftrightarrow X(a^{-1}z)$

PROOF $\sum_{n=0}^{\infty} a^n x[n] z^{-n} = \sum_{n=0}^{\infty} x[n] (a^{-1}z)^{-n} = X(a^{-1}z)$ ✓

MULT. BY n: $n\,x[n] \longleftrightarrow -z \dfrac{dX(z)}{dz}$

PROOF $-z \frac{d}{dz} \sum_{n=0}^{\infty} x[n] z^{-n} = \cancel{-} z \sum_{n=1}^{\infty} \cancel{-}n\,x[n] z^{-n-1} = \sum_{n=0}^{\infty} n\,x[n] z^{-n}$ ✓

EXAMPLE 1 $n\,u[n] \longleftrightarrow -z \frac{d}{dz} \frac{z}{z-1}$

$$= -z \frac{(z-1)-z}{(z-1)^2}$$

$$= \frac{z}{(z-1)^2}$$ ✓

EXAMPLE 2 $n a^n u[n] \longleftrightarrow \dfrac{a^{-1}z}{(a^{-1}z-1)^2} = \dfrac{az}{(z-a)^2}$ ✓

DELAY BY $c \geq 0$: $x[n-c]\,u[n-c] \longleftrightarrow z^{-c} X(z)$

PROOF $z^{-c} X(z) = \sum_{n=0}^{\infty} \dfrac{x[n]}{z^{c+n}}$

POWER OF z^{-1}	c	$c+1$	$c+2$	$c+3$	...
ITS COEFFICIENT	$x[0]$	$x[1]$	$x[2]$	$x[3]$	

EXAMPLE $u[n] - u[n-c] \longleftrightarrow \dfrac{z}{z-1} - \dfrac{z}{z-1} z^{-c} = \dfrac{z - z^{1-c}}{z-1}$
$c > 0$

$$= \frac{z^c - 1}{z^{c-1}(z-1)}$$

ADVANCE BY $c \geq 0$: $x[n+c]\,u[n] \longleftrightarrow z^c \left[X(z) - X_c(z) \right]$

$$X_c(z) = \sum_{n=0}^{c-1} x[n] z^{-n}$$

CONVOLUTION: $x[n] * v[n] = X(z)\,V(z)$

INVERSE z - TRANSFORM

- $x[n] = \frac{1}{2\pi j} \oint X(z)\, z^{n-1}\, dz$

  EVALUATED ALONG CONTOUR WHICH IS A COUNTER CLOCKWISE CIRCLE IN TRANFORM'S REGION OF CONVERGENCE

  } DO NOT USE THIS FORMULA!

- USE TABLES, PROPERTIES (JUST LIKE LAPLACE)
- WHEN $X(z) = \dfrac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}$, $M \le N$

→ USE PARTIAL FRACTIONS AND TABLE (JUST LIKE LAPLACE)

OTHER METHODS

- PERFORM THE LONG DIVISION AND USE "PLACEHOLDER" IDEA

  $X(z) = B(z)/A(z)$ → DIVIDE $A(z)$ INTO $B(z)$ USING LONG DIVISION
  → OBTAIN POWER SERIES IN z^{-1}

EXAMPLE

$$F(z) = \frac{2z^2 - 3z}{z^2 - 3z + 2}$$

$$\begin{array}{r|l} & 2 + 3z^{-1} + 5z^{-2} + \cdots \\ \hline z^2 - 3z + 2 & 2z^2 - 3z \\ & \underline{2z^2 - 6z + 4} \\ & 3z - 4 \\ & \underline{3z - 9 + 6z^{-1}} \\ & 5 - 6z^{-1} \\ & \ddots \end{array}$$

SO, $f[n] = 2, 3, 5, \ldots$

$$\frac{F(z)}{z} = \frac{2z-3}{(z-1)(z-2)} = \frac{1}{z-1} + \frac{1}{z-2}$$

$$F(z) = \frac{z}{z-1} + \frac{z}{z-2}$$

$$f[n] = 1 + 2^n, \; n \ge 0$$

- MATLAB:

```
>> b=[2 -3 0];
>> a=[1 -3 2];
>> f=dimpulse(b,a,10);
>> f'

ans =

     2     3     5     9    17    33    65   129   257   513
```

b: NUM. COEFFS. a: DEN. COEFFS. 10: # OF VALUES, N (0.. N-1)

$$f[n] \longleftrightarrow F(z)$$
$$f[n] * \delta[n] = f[n] \qquad F(z)\cdot 1 = F(z)$$

EIGENFUNCTIONS AND SOLVING DIFF. EQNS.

$$e^{\lambda t} \longrightarrow \boxed{\text{C-T LTI} \atop h(t)} \longrightarrow H(\lambda)\, e^{\lambda t}$$

$$\lambda^n \longrightarrow \boxed{\text{D-T LTI} \atop h[n]} \longrightarrow H(\lambda)\, \lambda^n$$

(λ COMPLEX)

$$Y(z) = H(z)X(z) = H(z)\frac{z}{z-\lambda} = \frac{Az}{z-\lambda} + \underbrace{\sim\sim\sim}_{\substack{\text{TRANSIENT,} \\ \text{DEPENDS ON} \\ \text{POLES OF } H(z)}}$$

$$A = \left[(z-\lambda)\frac{Y(z)}{z}\right]_{z=\lambda} = H(\lambda)$$

$p_i^n,\ n p_i^n$

$$\frac{d^N y(t)}{dt^N} + a_{N-1}\frac{d^{N-1}y(t)}{dt^{N-1}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t) = 0$$

GUESS THAT $y(t) = e^{\lambda t}$ IS A SOLUTION AND SUBSTITUTE:

$$\lambda^N e^{\lambda t} + a_{N-1}\lambda^{N-1}e^{\lambda t} + \cdots + a_1\lambda e^{\lambda t} + a_0 e^{\lambda t} = 0$$

DIVIDE BY $e^{\lambda t}$:

$$\lambda^N + a_{N-1}\lambda^{N-1} + \cdots + a_1\lambda + a_0 = 0 \quad (\star)$$

"CHARACTERISTIC POLYNOMIAL"

$e^{\lambda t}$ IS A SOLUTION IF AND ONLY IF λ IS A ROOT OF $(\star)$.

<u>EXAMPLE 1</u> $\quad \frac{dy}{dt} = ay \rightsquigarrow \underset{\lambda - a = 0}{\text{CHAR. EQN.}} \rightsquigarrow \underset{\lambda = a}{\text{ROOTS}} \rightsquigarrow \underset{y(t) = Ce^{\lambda t}}{\text{SOLN.}}$

<u>EXAMPLE 2</u> $\quad \frac{d^2y}{dt^2} - 5\frac{dy}{dt} + 6y \rightsquigarrow \lambda^2 - 5\lambda + 6 \rightsquigarrow \lambda = 2, 3 \rightsquigarrow y(t) = c_1 e^{2t} + c_2 e^{3t}$

<u>NOTE:</u> REPEATED ROOTS $\rightsquigarrow e^{\lambda t},\ te^{\lambda t},\ t^2 e^{\lambda t}, \ldots$

SOLVING LINEAR DIFFERENCE EQUATIONS

$$y[n+N] + a_{N-1}\, y[n+N-1] + \cdots + a_1\, y[n+1] + a_0\, y[n] = 0$$

GUESS THAT $y[n] = \lambda^n$ IS A SOLUTION AND SUBSTITUTE:

$$\lambda^{n+N} + a_{N-1}\, \lambda^{n+N-1} + \cdots + a_1\, \lambda^{n+1} + a_0\, \lambda^n = 0$$

DIVIDE BY λ^n:

$$\lambda^N + a_{N-1}\, \lambda^{N-1} + \cdots + a_1 \lambda + a_0 = 0 \quad (\star)$$

"CHARACTERISTIC POLYNOMIAL"

λ^n IS A SOLUTION IF AND ONLY IF λ IS A ROOT OF $(\star)$

EXAMPLE 1

$$y[n+1] = (1+r)\, y[n]$$

CHARACTERISTIC POLYNOMIAL: $\lambda - (1+r) = 0$

ROOTS: $\lambda = 1+r$

GENL. SOLN: $y[n] = C\,(1+r)^n$

(C) DETERMINED FROM INITIAL CONDITIONS

EXAMPLE 2

$$y[n+2] - 3y[n+1] + 2y[n] = 0$$

$$\rightarrow \lambda^2 - 3\lambda + 2 = 0 \rightarrow \lambda = 1, 2 \rightarrow y[n] = c_1 + c_2\, 2^n$$

EXAMPLE 3 (FIBONACCI SEQUENCE)

$$y[n+2] = y[n+1] + y[n], \qquad y[1] = y[2] = 1$$

CHAR. EQN: $\lambda^2 - \lambda - 1 = 0$

ROOTS: $\lambda_{1,2} = \dfrac{1 \pm \sqrt{1+4}}{2} \rightarrow \lambda_1 \approx 1.618,\ \lambda_2 \approx -0.618$

FORM OF SOLN: $y[n] = A\lambda_1^n + B\lambda_2^n$

$$\begin{aligned} 1 &= A\lambda_1 + B\lambda_2 \\ 1 &= A\lambda_1^2 + B\lambda_2^2 \end{aligned} \Rightarrow A = -B = \frac{1}{\sqrt{5}}$$

z-TRANSFORM SOLUTION OF DIFFERENCE EQUATIONS

- z-TRANSFORM IS BASIS FOR ALMOST "AUTOMATIC" SOLUTION OF DIFFERENCE EQNS.

- KEY IS TIME-ADVANCE PROPERTY

$$y[n+1] \longleftrightarrow z\,Y(z) - z\,y[0]$$

ITERATING...

$$\begin{aligned}
y[n] &\longleftrightarrow Y(z) \\
y[n+1] &\longleftrightarrow z\,Y(z) - z\,y[0] \\
y[n+2] &\longleftrightarrow z^2\,Y(z) - z^2\,y[0] - z\,y[1] \\
y[n+3] &\longleftrightarrow z^3\,Y(z) - z^3\,y[0] - z^2\,y[1] - z\,y[2] \\
&\vdots
\end{aligned}$$

- CONSIDER THE GENERAL CASE

$$y[n+N] + a_{N-1}\,y[n+N-1] + \cdots + a_1\,y[n+1] + a_0\,y[n] = g[n]$$

FIRST ASSUME THAT INIT. CONDITIONS $y[0] = y[1] = \cdots = y[N-1] = 0$

$$z^N\,Y(z) + a_{N-1}\,z^{N-1}\,Y(z) + \cdots + a_1 z\,Y(z) + a_1\,Y(z) = G(z)$$

$$\left(z^N + a_{N-1}\,z^{N-1} + \cdots + a_1 z + a_0\right) Y(z) = G(z)$$

$$Y(z) = \frac{G(z)}{z^N + a_{N-1}\,z^{N-1} + \cdots + a_1 z + a_0}$$

TO SOLVE:

1. FIND TRANSFORM OF $g[n]$
2. INVERT THE TRANSFORM TO FIND $y[n]$

NOTE: IF INITIAL CONDITIONS NON-ZERO, THEY MUST BE INCORPORATED $\longrightarrow$ MERELY CHANGES R.H.S. $G(z)$

EXAMPLES

EXAMPLE 1

$$y[n+1] - y[n] = 0 , \quad y[0] = 1$$

$$zY(z) - zy[0] - Y(z) = 0$$

$$Y(z) = \frac{z}{z-1} \longleftrightarrow y[n] = u[n]$$

EXAMPLE 2

$$y[n+1] + 2y[n] = 4^n , \quad y[0] = 0$$

$$zY(z) + 2Y(z) = \frac{z}{z-4}$$

$$Y(z) = \frac{z}{(z+2)(z-4)}$$

$$\frac{z}{(z+2)(z-4)} = \frac{1/3}{z+2} + \frac{2/3}{z-4} \longleftrightarrow \frac{1}{3}(-2)^{n-1} + \frac{2}{3}(4)^{n-1} , \quad n \geq 1$$

"TRANSIENTS" AT NATURAL FREQS. OF SYSTEM

NOTE: $$H(z) = \frac{1}{z+2} X(z)$$

$$H(4) = \frac{1}{4+2} = \frac{1}{6}$$

INPUT $4^n \longrightarrow$ S-S OUTPUT $H(4) \cdot 4^n = \frac{1}{6} 4^n = \frac{2}{3} \cdot \frac{1}{4} 4^n$ ✓

IN GENERAL

GIVEN $\sum_{k=0}^{N} a_k y[n+k] = \sum_{\ell=0}^{M} b_\ell x[n+\ell]$

ZSR IS INVERSE XFORM OF $H(z)X(z)$ WHERE

$$H(z) = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0}$$

ZIR: FIND POLES AND THEN $y[n] = \sum_{k=1}^{N} c_k p_k^N$ — NON REPEATED

NOTES: — REPEATED ROOTS $c N p_k^N$

— IF $|p_k| < 1$, ZIR DIES OUT, SYSTEM IS STABLE